Principles of Mathematics for Voting Systems

Lecture Notes

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Contents

1 Introduction **5**

2 The problem of apportionment **7**

2.1 Two basic approaches: Hamilton and D'Hondt 10

3 Some notions of Coalition Theory **15**

3.1 Cooperative games and voting games 15

3.2 Power indices 17

Chapter 1

Introduction

These short lecture notes intend to provide some notions on the problem of seat apportionment in voting systems, or electoral systems. I synthesized some of the main concepts of a theory which is definitely wider and more complex. Some related issues such as turnout, apportionment paradoxes and electoral systems which are not based on proportional representation have been left out of the picture for the sake of brevity.

In the first Chapter I analyzed the basic apportionment criteria and methods and exposed some simple cases.

In Chapter 2 I outlined some preliminary notions of Cooperative Game Theory and subsequently the application to weighted voting games with quota was introduced together with some numerical examples.

Because there are not so many academic courses on this subject, and coherently it is hard to find many textbooks treating the Mathematics of Electoral Systems in a complete way, I can just cite some references for future reading and future development. The first book which was completely devoted to this topic was written by Balinski and Young [2] (first edition in 1982, second edition in 2001), and this volume is considered as the seminal study on apportionment. Furthermore, a very good overview can be found in the book by Hodge and Klima (the first edition was released in 2005, the second one in 2018) [8]. Other contributions which are closer to my research activity are [4, 5, 6].

Quite differently, a lot of literature exists on Cooperative Games, which is applied to many other disciplines. The interested readers can expand their knowledge on Games by reading Gonzalez-Diaz *et al.* [7] or Owen [10], just to cite 2 particularly clear-cut textbooks. It can be also very interesting to look at the original papers on power indices by Banzhaf [3], Myerson [9] and Shapley [11]. For further suggestions, I encourage any interested scholar to contact me (Arsen.Palestini@uniroma.it).

Chapter 2

The problem of apportionment

The main issue can be summarized by a question: given the results of an election, which is the best way to allot seats amongst parties which obtained different amounts of votes? When we say 'the best way', we mean the way which best represents the electoral power of parties, or equivalently, the way which allows the smallest number of distortions and paradoxes that may occur in the apportionment process.

We are going to focus our attention on the scenario in which there is only one electoral district. It is easy to note that when districts are more than one, the complexity of the problem dramatically increases (see for example [5, 6] for an exhaustive discussion). First, some notation needs to be established:

- Let $N \geq 2$ be the number of involved parties. Call $\mathbf{v} = (v_1, \ldots, v_N)$ the vector of votes, i.e. $v_i \geq 0$ is the number of votes gained by the i-th party in the election. Intuitively, we can assume that v_i is a non-negative integer number for all $i = 1, \ldots, N$. The total number of votes is given by $v^T = v_1 + \cdots + v_N$.

- Call $\mathbf{s} = (s_1, \ldots, s_N)$ the vector of seats that will be assigned to the parties which are running for election. The total sum of the seats is typically fixed *ex ante*, in compliance with the institutional structure (a Parliament or a Senate, for example): $s^T = s_1 + \cdots + s_N$.

- The **Hare quota**[1] h_i of the i-th party is defined[2] as follows, for all parties $i = 1, \ldots, N$:
$$h_i = \frac{v_i \cdot s^T}{v^T}. \tag{2.0.1}$$

- For each party $i = 1, \ldots, N$, call the **Hare maximum** h_i^U and the **Hare minimum** h_i^D the two values which are respectively obtained by rounding up

[1]Note that, for simplicity, we are going to call Hare quota the quantity that sometimes is denoted by the ratio Votes/Quota.

[2]Sir **Thomas Hare** (1806-1891) was a British politician and lawyer.

and rounding down the Hare quota of the i-th party. If h_i is an integer number, obviously they all coincide: $h_i = h_i^U = h_i^D$.

In the next Example, we are going to take into account a very simple scenario with 4 parties among which 5 seats have to be allotted.

Example 1. *Suppose that* 4 *parties* $(A, \ B, \ C, \ D)$ *are running for election and that the total number of seats to win are* 5*. The votes gained by the parties and their Hare quotas are collected in the following table:*

Parties	A	B	C	D
Votes	38	22	7	43
Hare quotas	1.72	1	0.31	1.95
Hare maxima	2	1	1	2
Hare minima	1	1	0	1

The total number of votes is $v^T = 38+22+7+43 = 110$*. The Hare quotas are easily calculated:*

$$h_A = \frac{38 \cdot 5}{110} = 1.72, \qquad h_B = \frac{22 \cdot 5}{110} = 1, \qquad h_C = \frac{7 \cdot 5}{110} = 0.31, \qquad h_D = \frac{43 \cdot 5}{110} = 1.95.$$

Party B finds itself in the easiest situation, because its Hare quota is integer and corresponds to 1 *seat. In the remianing cases, the seats to be allotted are uncertain: there will be one seat each for all parties except party C, in compliance with the Hare minima, but there are still* 2 *seats to be assigned. On the other hand, if all the parties gained the seats established by the Hare maxima, their total seats would be* 6*, one more than the actual seats of the Parliament. This means that some parties will obtain a number of seats which is equal to their Hare maxima and some other parties will obtain the number of seats which correspond to their Hare minima.*

Namely, if all the seats were assigned in compliance with Hare minima, there would be 2 *remaining seats to be assigned. On the other hand, if all the seats were assigned in compliance with Hare maxima, one more seat in the Parliament would be necessary.*

It is intuitive to see that some criteria are necessary to allot the seats. There are many criteria which can be established and that have to be verified to achieve a fair apportionment. We are going to introduce the following 3 basic criteria:

- **Monotonicity:** given any pair of parties A and B, if A gains more votes than B, then B cannot gain more seats than A:

$$v_A \geq v_B \qquad \Longleftrightarrow \qquad s_A \geq s_B.$$

- **Hare maximum criterion:** each party cannot obtain a number of seats which is larger than its Hare maximum.

- **Hare minimum criterion:** each party cannot obtain a number of seats which is smaller than its Hare minimum.

The 2 criteria relying on Hare quotas can be reformulated as follows: given a party $i \in \{1, \ldots, N\}$, we must have that

$$h_i^D \leq s_i \leq h_i^U.$$

There are only 2 apportionments that satisfy the above criteria in the last Example. They are:

Parties	A	B	C	D
Votes	38	22	7	43
Seats	2	1	0	2

and

Parties	A	B	C	D
Votes	38	22	7	43
Seats	1	1	1	2

Remark 2. *In the last Example we did not take into account any electoral threshold, that is each party is entitled to the assignment of seats, regardless of the amount of its votes. If we calculate the shares of each party, as is usually computed in the ballots, our table becomes as follows:*

Parties	*A*	*B*	*C*	*D*
Votes	38	22	7	43
Shares	$34,54\%$	20%	$6,36\%$	$39,09\%$

The presence of an electoral threshold potentially allows us to solve the apportionment problem immediately. Clearly, it depends on the threshold value. In fact, if the threshold were 10%*, party* C *would not be entitled to the seat assignment, because its share is lower than the threshold, consequently* 2 *seats would be assigned to parties* A *and* D *and* 1 *seat would be assigned to* B*.*

On the other hand, if the threshold were 4% *or* 5%*, which are typical electoral threshold levels,* C *would participate in the apportionment, because its share exceeds the threshold, and the problem would not be solved.*

The discussion on apportionment criteria can be widely extended, and many other criteria, going beyond the scope of the present lecture notes, have been conceived and proposed. Over the years, some possible paradoxes have been identified (the first one is the well-known *Condorcet's paradox*, which dates back to the 18^{th} century, and which was independently discovered by Reverend Lewis Carroll, author of *Alice's Adventures in Wonderland*).

It is interesting to briefly recall Condorcet's paradox, keeping in mind that voters can express multiple preferences, thus giving a score to each party, or a kind of ranking among them. Suppose that 3 voters have to rank 3 parties: A, B and C. If the voters select the following ranking for their preferences (see the Table below):

Parties	A	B	C
Voter 1	1	2	3
Voter 2	3	1	2
Voter 3	2	3	1

this means that the preferences are cyclical, i.e. every party achieved the same total score and there is a draw, so no party wins.

Another crucial result, known as *Arrow's impossibility Theorem* [1], was obtained by Kenneth Arrow in 1950, proving the impossibility of establishing any apportionment which verified a sequence of criteria. To summarize, such results highlight that in most cases it's mathematically impossible to construct apportionments which verify all the necessary criteria. In the next Section we are going to analyze two standard apportionment procedures.

2.1 Two basic approaches: Hamilton and D'Hondt

The **Proportional Representation Method** was first proposed and subsequently improved by several scholars since the 18th century.

The first method we are going to describe was conceived by **Alexander Hamilton** (1755-1804), who was an American economist, lawyer and politician. His method, which is called the **Largest Remainders Method**[3] was adopted in the USA between 1852 and 1911. Basically, it is based on the Hare minimum of the involved parties, although sometimes an alternative quota is employed, i.e. the *Droop quota*[4].

Initially, each party wins a number of seats which corresponds to its Hare minimum, and subsequently, we subtract the above number of seats from the Hare quotas, thus obtaining the remainders, which are numbers smaller than 1. Then the remaining seats are assigned to parties having the largest remainders, until the last seat.

To clarify the procedure, we consider Example 1 once more. The table of votes, seats and remainders is:

[3]This method is sometimes called Hare-Niemeyer's Method or Vinton's Method.

[4]When using the Droop quota, the ratio $\frac{v^T}{s^T}$ is replaced by the quantity $1 + \frac{v^T}{1+s^T}$.

Parties	A	B	C	D
Votes	38	22	7	43
Hare quotas	1.72	1	0.31	1.95
Seats assigned according to Hare minimum	1	1	0	1
Remainders	0.72	0	0.31	0.95
Seats assigned by Hamilton's Method	1	0	0	1
Total seats	2	1	0	2

As has already been pointed out, this seat apportionment verifies the 3 fundamental criteria.

The second approach we are going to introduce is the **Highest Averages Method**, which was proposed by **Victor Joseph Auguste D'Hondt** (1841-1901), a Belgian lawyer and professor of law, almost one century after Hamilton.

This procedure does not require the calculation of the Hare quotas, because it is based on divisions. Each party's amount of votes has to be divided by the first integer numbers such as 1, 2, 3, and so on. The seats are then assigned by taking the highest values obtained by such divisions. We can apply such a method to Example 1 again to achieve a seat apportionment. Therefore, we construct a table with the results of all the necessary divisions:

Parties	A	B	C	D
Votes	38	22	7	43
Votes divided by 1	38	22	7	43
Votes divided by 2	19.5	11	3.5	21.5
Votes divided by 3	$12.\overline{6}$	$7.\overline{3}$	$2.\overline{3}$	$14.\overline{3}$
Seats assigned by D'Hondt's Method	2	1	0	2

Note that the resulting apportionment is the same as the one we obtained by the largest Remainders Method. The seats have been assigned in such a way because the ranking of remainders is the following:

- 43 (1 seat for D);
- 38 (1 seat for A);
- 22 (1 seat for B);
- 21.5 (1 seat for D);
- 19.5 (1 seat for A).

It is interesting to note what would happen if the seats to be apportioned were more than 5. Party C would have only gained the 10^{th} seat, because there are 9 values which are greater than 7.

What follows is a further example where the 2 methods we illustrated produce different seat apportionments.

Example 3. *Suppose that* 5 *parties (A, B, C, D, E) are running for election and that the total number of seats to win are* 13*. The votes gained by the parties are collected in the following table:*

Parties	*A*	*B*	*C*	*D*	*E*
Votes	71	24	49	26	50

The total number of votes is $v^T = 71 + 24 + 49 + 26 + 50 = 220$*. We are going to apply both methods, beginning from Hamilton's Method.*

Parties	*A*	*B*	*C*	*D*	*E*
Votes	71	24	49	26	50
Hare quotas	4.195	1.418	2.895	1.536	2.954
Seats assigned according to Hare minimum	4	1	2	1	2
Remainders	0.195	0.418	0.895	0.536	0.954
Seats assigned by Hamilton's Method	0	0	1	1	1
Total seats	4	1	3	2	3

On the other hand, if we employ D'Hondt's Method we have:

Parties	*A*	*B*	*C*	*D*	*E*
Votes	71	24	49	26	50
Votes divided by 1	71	24	49	26	50
Votes divided by 2	35.5	12	24.5	13	25
Votes divided by 3	$23.\overline{6}$	8	$16.\overline{3}$	$8.\overline{6}$	$16.\overline{6}$
Votes divided by 4	17.75	6	12.25	6.5	12.5
Vots divided by 5	14.2	4.8	9.8	5.2	10
Total seats assigned by D'Hondt's Method	5	1	3	1	3

As can be easily seen, the 2 *methods generate* 2 *different apportionments: applying Hamilton's Method yields* $(4, 1, 3, 2, 3)$*, whereas D'Hondt's Method leads to the assignment* $(5, 1, 3, 1, 3)$*, thus giving more seats to party A, which is the major party, and less seats to party D, which is the fourth one in the ranking.*

Now we should check whether the apportionments we achieved in the previous Example actually verify the basic criteria. Monotonicity is clearly respected: it is immediate to note that the orderings associated to both apportionments reproduce the orderings of votes of the parties. On the other hand, Hare maximum and Hare minimum criteria are satisfied by both of them as well: the number of seats obtained is their Hare maximum for parties C and E in both apportionments, whereas B always gets its Hare minimum. A obtains Hare minimum using Hamilton's Method and Hare maximum using D'Hondt's Method. Conversely, D obtains Hare maximum when using Hamilton's Method and Hare minimum when using D'Hondt's Method.

This leads to a further problem: is there a way to select the apportionments based on other aspects? In other words, after guaranteeing that the 3 basic criteria are verified, is there another criterion which makes us prefer one apportionment?

The answer is yes, we can employ other criteria, for example the ones which rely on the simple concept of *coalitions*. Basically, we must assume that such parties obtain their seats in the Parliament to subsequently form coalitions and possibly a Government having the majority of seats. We know that in most democratic systems a Government is formed when some parties establish an agreement based on ideological similarities or else. In the following Chapter, we are going to investigate some notions of Coalition Theory, under very easy assumptions. For further reading, see for example[4, 10].

Chapter 3

Some notions of Coalition Theory

3.1 Cooperative games and voting games

Now it's necessary to exactly characterize a type of *voting game* which is based on coalitions and on coalitional power. We need some simple definitions borrowed from Cooperative Game Theory (the related theory can be found in many textbooks for graduate students, such as [7, 10]).

Definition 4. *Given a set of N players $\mathcal{N} = \{1, 2, \ldots, N\}$, a* ***cooperative game (or TU-game)*** *is a pair $(\mathcal{N},\ v)$, where $v : 2^{\mathcal{N}} \longrightarrow \mathbb{R}$ such that $v(S)$ is the value of the coalition $S \in 2^{\mathcal{N}}$, and $v(\emptyset) = 0$.*

Cooperative games can be applied to a large number of economic, financial and political scenarios and settings. We are just going to focus on the aspects which are connected to voting power and seat apportionment. The easiest game that can be defined is a game in which the value of each coalition $S \in 2^{\mathcal{N}}$ can be equal to 0 or to 1. Basically, if $v(S) = 1$, S is a *winning coalition*, whereas it is a *losing coalition* otherwise. Suppose that each player has an endowment, in the form of a nonnegative integer number: for every $i = 1, \ldots, N$, the i-th player (or party) has the endowment[1] $w_i \geq 0$.

Definition 5. *Given a set of N players $\mathcal{N} = \{1, 2, \ldots, N\}$, a nonnegative vector $\mathbf{w} = (w_1, \ldots, w_N)$ and a number $q > 0$ such that*

$$0 < q \leq \sum_{i=1}^{N} w_i,$$

[1] Such a number can either denote the number of votes or the number of seats. If it expresses the number of seats, we are assuming that the apportionment has already been established.

a ***weighted voting game (or weighted majority game)*** *is a pair* $(\mathcal{N},\ w)$ *such that*

$$w(S) = \begin{cases} 1 & \text{if } \sum_{i\in S} w_i \geq q, \\ \\ 0 & \text{otherwise} \end{cases} .$$

The Definition 5 introduces a game where there are winning and losing coalitions based on the quota q. If the sum of endowments of the parties belonging to a coalition S exceeds the given quota q, S is a winning coalition. Clearly, the most intuitive choice for quota q is the absolute majority, that is half of the sum of all endowments of the parties. For example, in the Italian Chamber of Deputies the members are 630, hence the quota for simple majority is 316 (if the total number of seats is even, as in this case, usually the quota is increased by 1 with respect to the half, whereas if the total number is odd, typically the half is rounded up).

Example 6. *Consider* 4 *parties in a Parliament having the following numbers of seats. We are going to identify all the winning coalitions.*

Parties	A	B	C	D
Seats	35	64	18	43

Since the total number of seats is 160*, we choose* $q = 81$ *as the quota. The related game* $(\{A,\ B, C, D\}\,,\ w)$ *has the following values:*

$$w(\{A,B,C,D\}) = 1, \qquad \textit{because} \ \ 35+64+18+43 = 160 > 81.$$

$$w(\{A,B,C\}) = 1, \qquad \textit{because} \ \ 35+64+18 = 117 > 81.$$

$$w(\{B,C,D\}) = 1, \qquad \textit{because} \ \ 64+18+43 = 125 > 81.$$

$$w(\{A,C,D\}) = 1, \qquad \textit{because} \ \ 35+18+43 = 96 > 81.$$

$$w(\{A,B,D\}) = 1, \qquad \textit{because} \ \ 35+64+43 = 142 > 81.$$

$$w(\{A,B\}) = 1, \qquad \textit{because} \ \ 35+64 = 99 > 81.$$

$$w(\{B,C\}) = 1, \qquad \textit{because} \ \ 64+18 = 82 > 81.$$

$$w(\{B,D\}) = 1, \qquad \textit{because} \ \ 64+43 = 107 > 81.$$

$$w(\{A,C\}) = 0, \qquad \textit{because} \ \ 35+18 = 53 < 81.$$

$$w(\{A,D\}) = 0, \qquad \textit{because} \ \ 35+43 = 78 < 81.$$

$$w(\{C,D\}) = 0, \qquad \textit{because} \ \ 18+43 = 61 < 81.$$

$$w(\{A\}) = 0, \qquad \textit{because} \ \ 35 < 81.$$

$$w(\{B\}) = 0, \qquad \textit{because} \ \ 64 < 81.$$

$$w(\{C\}) = 0, \qquad \textit{because} \;\; 18 < 81.$$

$$w(\{D\}) = 0, \qquad \textit{because} \;\; 43 < 81.$$

Finally, by Definition of TU-game, $w(\emptyset) = 0$.

Note that intuitively, we can 'count' the number of coalitions containing a given party to assess that party's power in forming winning coalitions. However, A, C and D participate in 5 winning coalitions, whereas B participates in 7 winning coalitions.

As can be noted, a coalition game has some particular characteristics: basically, A and C can form the same number of winning coalitions although the endowment of A is almost twice the endowment of C. Hence, the endowments reflect the actual power of parties only in a partial way. It is also interesting to see that if a higher quota were chosen, say 90, the coalition $\{B, C\}$ *would be losing, hence B would participate in just 6 winning coalitions, A and D in 5 winning coalitions as before, and C might win only in 4 coalitions. This means that the choice of the quota is relevant as well.*

Is there a precise measure of the players' power in the formation of the coailtions? There are several indicators of such a power, that will be introduced in the next Section. They are solution concepts of cooperative games, called *power indices.*

3.2 Power indices

In order to precisely assess the power of each party to contribute to the possible win of its coalition, we can employ the most common solution concepts, i.e the Shapley value[2], which was first introduced in [11] in 1953, or the Banzhaf value[3] of the game (see [3]). Consider an N-players game $(\mathcal{N},\ v)$. The Shapley value and the Banzhaf value are respectively defined as follows, for all assets $i = 1, \ldots, N$:

- **Shapley value:**

$$\Phi_i(v) = \sum_{i \in S,\ S \subseteq \mathcal{N}} \frac{(N - |S|)!(|S| - 1)!}{N!} \left[v(S) - v(S \setminus \{i\}) \right]. \tag{3.2.1}$$

- **Banzhaf value:**

$$\beta_i(v) = \frac{1}{2^{N-1}} \sum_{i \in S,\ S \subseteq \mathcal{N}} \left[v(S) - v(S \setminus \{i\}) \right]. \tag{3.2.2}$$

These values are the most famous and used ones and many axiomatizations of them have been constructed and published in literature. However, many other indices

[2] The Shapley value is sometimes indicated as Shapley-Shubik value.

[3] The Banzhaf value is sometimes indicated as the Banzhaf-Coleman value.

have been proposed (see [10]), also on more complex structures such as graphs. Basically, in a graph representing a political scenario, each party is a node of the graph, and the links connecting the pairs of nodes represent the feasibility of that agreement. In other words, if parties A and B are not connected by a link, they cannot form a coalition directly. Roger Myerson proposed a power index to be applied to graphs [9] in 1977, which was essentially a restriction of the Shapley value on the graph structures.

Before delving into the interpretation of such indices in Election Games, we can take a look at a very simple calculation of Shapley and Banzhaf indices in a 3-players game.

Example 7. *Consider the following simple game having $\{A, B, C\}$ as the set of players. The characteristic value function has the following values:*

$$v(\{A, B, C\}) = 1,$$

$$v(\{A, B\}) = 1, \qquad v(\{A, C\}) = 1, \qquad v(\{B, C\}) = 0,$$
$$v(\{A\}) = 0, \qquad v(\{B\}) = 0, \qquad v(\{C\}) = 0.$$

Note that in this particular case if we remove player A from the grand coalition $\{A, B, C\}$, the resulting coalition is losing. Players having this property are called ***veto players****.*

First, we calculate the components of the Shapley value:

$$\Phi_A(v) = \frac{(3-3)!(3-1)!}{3!}[v(\{A, B, C\}) - v(\{B, C\})]+$$
$$+\frac{(3-2)!(2-1)!}{3!}[v(\{A, B\}) - v(\{B\}) + v(\{A, C\}) - v(\{C\})]+$$
$$+\frac{(3-1)!(1-1)!}{3!}[v(\{A\}) - v(\emptyset)] = \frac{1}{3} + \frac{2}{6} + 0 = \frac{2}{3}.$$
$$\Phi_B(v) = \frac{(3-3)!(3-1)!}{3!}[v(\{A, B, C\}) - v(\{A, C\})]+$$
$$+\frac{(3-2)!(2-1)!}{3!}[v(\{A, B\}) - v(\{A\}) + v(\{B, C\}) - v(\{C\})]+$$
$$+\frac{(3-1)!(1-1)!}{3!}[v(\{B\}) - v(\emptyset)] = 0 + \frac{1}{6} + 0 = \frac{1}{6}.$$
$$\Phi_C(v) = \frac{(3-3)!(3-1)!}{3!}[v(\{A, B, C\}) - v(\{A, B\})]+$$
$$+\frac{(3-2)!(2-1)!}{3!}[v(\{A, C\}) - v(\{A\}) + v(\{B, C\}) - v(\{B\})]+$$

$$+\frac{(3-1)!(1-1)!}{3!}[v(\{C\})-v(\emptyset)] = 0+\frac{1}{6}+0=\frac{1}{6}.$$

Therefore, the Shapley value of the game $(\{A,B,C\},v)$ *is:*

$$\Phi(v) = \left(\frac{2}{3},\ \frac{1}{6},\ \frac{1}{6}\right).$$

Now we turn to the calculation of the Banzhaf value, one component at a time:

$$\beta_A(v) = \frac{1}{2^2}\left[v(\{A,B,C\}) - v(\{B,C\}) + \right.$$
$$+v(\{A,B\}) - v(\{B\}) + v(\{A,C\}) - v(\{C\}) + v(\{A\}) - v(\emptyset)] =$$
$$= \frac{1}{4}[1-0+1-0+1-0+0-0] = \frac{3}{4}.$$
$$\beta_B(v) = \frac{1}{2^2}\left[v(\{A,B,C\}) - v(\{A,C\}) + \right.$$
$$+v(\{A,B\}) - v(\{A\}) + v(\{B,C\}) - v(\{C\}) + v(\{B\}) - v(\emptyset)] =$$
$$= \frac{1}{4}[1-1+1-0+0-0+0-0] = \frac{1}{4}.$$
$$\beta_C(v) = \frac{1}{2^2}\left[v(\{A,B,C\}) - v(\{A,B\}) + \right.$$
$$+v(\{A,C\}) - v(\{A\}) + v(\{B,C\}) - v(\{B\}) + v(\{C\}) - v(\emptyset)] =$$
$$= \frac{1}{4}[1-1+1-0+0-0+0-0] = \frac{1}{4}.$$

Hence, the Banzhaf value of the game (whose sum of components exceeds the value of the grand coalition, differently from the Shapley valuee) is:

$$\beta(v) = \left(\frac{3}{4},\ \frac{1}{4},\ \frac{1}{4}\right).$$

Note that both power indices assign to parties B *and* C *the same power.*

How can such power indices play a role in the apportionment of seats in a Parliament? Basically, since they appropriately describe the coalitional power of parties, the parties having more power are more entitled to form coalitions, thereby guaranteeing the Government's stability. This means that it makes sense to assign more seats to a party which has more power in compliance with the Shapley or the Banzhaf index.

We are going to focus our attention on Example 3 again. We can remember that we obtained 2 different apportionments when we apply 2 distinct apportionment methods. Our aim is to investigate which apportionments may be induced by the power indices. Before facing the problem, note that the cooperative game has 5 players, thus meaning that the related computational cost is rather high. For this reason, when players are more than 4 or 5, it is highly recommended to use a mathematical tool or software to implement the calculations (either Matlab or Mathematica or R or Excel any other one).

Example 8. *We consider the* 5 *parties (A, B, C, D, E) that we took into account previously to calculate the Shapley and the Banzhaf indices. The Parliament seats to win are* 13*, whereas the votes gained by the parties:*

Parties	*A*	*B*	*C*	*D*	*E*
Votes	71	24	49	26	50

Since the total number of votes is 220*, the winning coalitions are the ones which collect at least* 111 *total votes. They can be enumerated:*

- 5***-parties winning coalitions:*** *the grand coalition* $\{A, B, C, D, E\}$.
- 4***-parties winning coalitions:*** $\{A, B, C, D\}$, $\{A, B, C, E\}$, $\{A, B, D, E\}$, $\{A, C, D, E\}$, $\{B, C, D, E\}$.
- 3***-parties winning coalitions:*** $\{A, B, C\}$, $\{A, B, E\}$, $\{A, C, D\}$, $\{A, D, E\}$, $\{A, B, D\}$, $\{A, C, E\}$, $\{B, C, E\}$, $\{C, D, E\}$.
- 2***-parties winning coalitions:*** $\{A, E\}$, $\{A, C\}$.
- 1***-party winning coalitions:*** *none.*

Now we turn to compute the Shapley value (in a synthetical way):

$$\Phi_A = \frac{0!4!}{5!} \cdot 0 + \frac{1!3!}{5!} \cdot 2 + \frac{2!2!}{5!} \cdot 6 + \frac{3!1!}{5!} \cdot 2 + \frac{4!0!}{5!} \cdot 0 = \frac{1}{10} + \frac{1}{5} + \frac{1}{10} = \frac{2}{5}.$$

$$\Phi_B = \frac{0!4!}{5!} \cdot 0 + \frac{1!3!}{5!} \cdot 0 + \frac{2!2!}{5!} \cdot 2 + \frac{3!1!}{5!} \cdot 0 + \frac{4!0!}{5!} \cdot 0 = \frac{1}{15}.$$

$$\Phi_C = \frac{0!4!}{5!} \cdot 0 + \frac{1!3!}{5!} \cdot 1 + \frac{2!2!}{5!} \cdot 4 + \frac{3!1!}{5!} \cdot 1 + \frac{4!0!}{5!} \cdot 0 = \frac{1}{20} + \frac{2}{15} + \frac{1}{20} = \frac{7}{30}.$$

$$\Phi_D = \frac{0!4!}{5!} \cdot 0 + \frac{1!3!}{5!} \cdot 0 + \frac{2!2!}{5!} \cdot 2 + \frac{3!1!}{5!} \cdot 0 + \frac{4!0!}{5!} \cdot 0 = \frac{1}{15}.$$

$$\Phi_E = \frac{0!4!}{5!} \cdot 0 + \frac{1!3!}{5!} \cdot 1 + \frac{2!2!}{5!} \cdot 4 + \frac{3!1!}{5!} \cdot 1 + \frac{4!0!}{5!} \cdot 0 = \frac{1}{20} + \frac{2}{15} + \frac{1}{20} = \frac{7}{30}.$$

So the Shapley value of this weighted voting game is

$$\Phi = \left(\frac{2}{5}, \frac{1}{15}, \frac{7}{30}, \frac{1}{15}, \frac{7}{30}\right).$$

On the other hand, the Banzhaf value is:

$$\beta_A = \frac{1}{2^4}[0+2+6+2+0] = \frac{5}{8}.$$

$$\beta_B = \frac{1}{2^4}[0+0+2+0+0] = \frac{1}{8}.$$

$$\beta_C = \frac{1}{2^4}[0+1+4+1+0] = \frac{3}{8}.$$

$$\beta_D = \frac{1}{2^4}[0+0+2+0+0] = \frac{1}{8}$$

$$\beta_E = \frac{1}{2^4}[0+1+4+1+0] = \frac{3}{8}.$$

So the Banzhaf value of this weighted voting game is

$$\beta = \left(\frac{5}{8}, \frac{1}{8}, \frac{3}{8}, \frac{1}{8}, \frac{3}{8}\right).$$

Finally, we are going to collect in a table the found indices and the apportionment which were obtained by applying Hamilton's Method and D'Hondt's method (rather than fractions, we are going to use the associated numbers or their approximations).

Parties	*A*	*B*	*C*	*D*	*E*
Votes	71	24	49	26	50
Seats assigned by Hamilton's Method	4	1	3	2	3
Seats assigned by D'Hondt's Method	5	1	3	1	3
Shapley index	0.4	0.066	0.233	0.066	0.233
Banzhaf index	0.625	0.125	0.375	0.125	0.375

It is worthwile to analyze this table. As can be immediately seen, both power indices exactly reproduce the apportionment achieved by D'Hondt's Method, providing the same ranking. Hamilton's Method deviates by assigning 1 *additional seat to* D*, but* D *has the same coalitional power as* B*. Hence, the apportionment induced by Hamilton's Method is less useful in terms of coalitions.*

Basically, we needed a further criterion to decide between the 2 *apportionment procedures. Since both power indices reproduce D'Hondt's solution, D'Hondt's solution turns out to be preferable.*

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