# A Fractional Stochastic Regularity Model

Daniele Angelini[a] Sergio Bianchi[a,b]

[a]MEMOTEF, University "La Sapienza" Rome, Italy

[b]Intl Affiliate, Dept. Finance and Risk Engineering, New York University NY, USA

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# Problem and Research Question

Starting from the Comte and Renault's Fractional Stochastic Volatility model (1998), Gatheral et al. introduced a new model based on the volatility roughness observations.

## Rough Fractional Stochastic Volatility (RFSV) model

$$\begin{cases} dS_t = \mu_t S_t dt + S_t \sigma_t dW_t \\ d\log(\sigma_t) = \alpha[m - \log(\sigma_t)]dt + \rho dB_t^H \end{cases} \tag{1}$$

$\log \sigma_t$ is modeled with a *fractional Ornstein-Uhlenbeck* (fOU) process driven by a $H \sim 0.1$.

# Is volatility really rough?

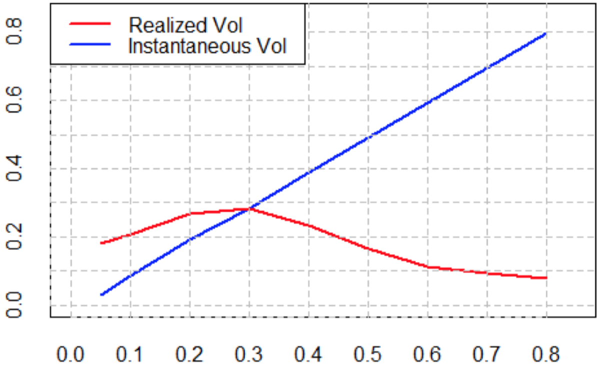


Figure: Hurst estimation for the realized volatility of stock price, estimated from fOU with different Hurst exponent. Source: Cont and Das (2022).

# Is volatility a good risk measure?

**Volatility $\sigma_t$**

tells us "*how much*" data are dispersed around a mean value.

**Regularity $H_t$**

tells us "*how*" and "*how much*" data get dispersed.

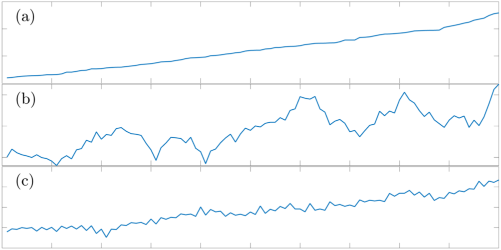


Figure: Different paths with same volatility. Source: Bianchi et al. (2022).

# Multifractal behaviour of price paths

Brandi and Di Matteo (2022) show that a negative correlation exists between the multiscaling proxy of the prices and the global Hurst exponent of the volatility for many market indexes.

This empirical evidence led us to combine a multiscaling model for prices with a fractional process for the volatility dynamics.

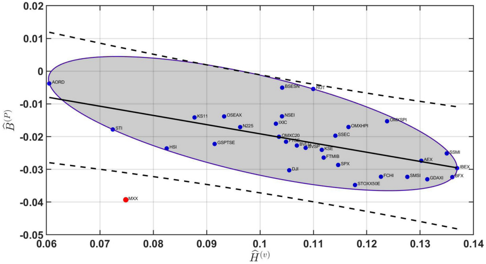


Figure: Source: Brandi and Di Matteo (2022).

# Contributions

- We introduce the *Fractional Stochastic Regularity Model* (FSRM) where the stock price $S_t$ is modeled by a *Multifractional Process with Random Exponent* (MPRE) and his Hurst exponent, modeled by a *fractional Ornstein-Uhlenbeck* (fOU), replace the $\log \sigma_t$.
- We show that highly nonlinear biases, related to the choice of some estimation parameters and to the estimator's own variance, can affect the estimates of the global Hurst exponent of the $\log \sigma_t$ and can lead to artificial roughness.

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# Fractional Brownian motion

Moving beyond the Brownian motion paradigm, Mandelbrot and Von Ness introduced tha fractional Brownian motion with a Hurst exponent $H \in (0, 1)$:

$$B_t^{H,C} = \frac{CV_H^{1/2}}{\Gamma(H + 1/2)} \int_{\mathbb{R}} \left( (t - s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) dW_s \tag{2}$$

where $C$ is a parameter, $V_H = \Gamma(2H + 1) \sin(\pi H)$ and the measure $dW$ is Gaussian and independently scattered.

We can discretized it in the $X_j^H$, with $j \in [\![1, n]\!]$, on the interval $t \in [0, 1]$. His variance is

$$\sigma^2(n) = Var(X_{j+1}^H - X_j^H) = C^2 n^{-2H} \implies \log \sigma(n) = \log C - H \log n. \tag{3}$$

# Multifractional Processes with Random Exponent (MPRE)

- Following the evidences observed by Brandi and Di Matteo about multifractality of prices, we would to model $S_t$ as a MPRE.
- Given the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\mathbb{R}}, \mathbb{P})$, let $\{W_s, s \in \mathbb{R}\}$ denote the standard Brownian motion w.r.t. $\mathcal{F}_t$ and, for each $t \geq 0$, let $g_s(t)$ be $\mathcal{F}_s$-adapted such that $g_s(t) = 0$ for $s > t$ with $\int_{-\infty}^{t} |g_s(t)|^2 ds < \infty$ for all $t$.
- The MPRE admits the following moving average representation

$$Z_t = \int_{-\infty}^{t} g_s(t) dW_s. \tag{4}$$

If we choose a $g_s(t) := C[(t-s)_+^{H_s-1/2} - (-s)_+^{H_s-1/2}]$, under several conditions of continuity for $H_t$ and local boundedness (see Loboda et al. (2021)), we obtain a special case of MPRE

$$B_t^{H_t,C} = C \int_{-\infty}^{t} \left[ (t-s)_+^{H_s-1/2} - (-s)_+^{H_s-1/2} \right] dW_s \tag{5}$$

driven by a stochastic function $H_t$ as *fractional Brownian motion* or *fractional Ornstein-Uhlenbeck*.

# Relation between $H_t$ and $\log \sigma_t$

- Under a mild technical condition, at any time $t_0$ the pointwise Hölder exponent of $B_{t_0}^{H_{t_0},C}$ equals $H_{t_0}$ almost surely and $B_t^{H_t,C}$ fulfills

$$\lim_{\epsilon \to 0^+} \left( \frac{B_{t+\epsilon u}^{H_{t+\epsilon u},C} - B_t^{H_t,C}}{\epsilon^{H_t}} \right)_{u \in \mathbb{R}^+} \stackrel{d}{=} \left( B_u^{H_t,C} \right)_{u \in \mathbb{R}^+}. \tag{6}$$

In the neighborhood of any time $t$, $B_t^{H_t,C}$ behaves like a fBm with $H_t$ parameter.

- If the fBm tangent to the MPRE has a constant unit time variance equal to $C^2$, at each point $t$

$$\log \sigma_t(n) = \log C - H_t \log n \implies$$

$$H_t = -\frac{1}{\log n} \log \sigma_t + \frac{\log C}{\log n}. \tag{7}$$

# Fractional Stochastic Regularity Model

In the *Rough Fractional Stochastic Volatility* model the $\log \sigma_t$ follows a fOU process $X_t$, $t \in [0, T]$, whose the unique pathwise solution is

$$X_t = m + \rho \int_{-\infty}^{t} e^{-\alpha(t-s)} dB_s^H \tag{8}$$

Rewriting the linear relationship in (7) as $H_t = p_1 \log \sigma_t + p_2$ we can model the Hurst parameter with a fOU

$$H_t = m' + \rho' \int_{-\infty}^{t} e^{-\alpha(t-s)} dB_s^H \tag{9}$$

with $m' = mp_1 + p_2$ and $\rho' = \rho p_1$.

Finally we can introduce our model:

## Fractional Stochastic Regularity Model

$$\begin{cases} S_t = B_t^{H_t, C} \\ H_t = m' + \rho' \int_{-\infty}^{t} e^{-\alpha(t-s)} dB_s^H, \end{cases} \tag{10}$$

where $S_t$ is modeled by a MPRE driven by stochastic Hurst-Hölder exponent, which is modeled with a fOU.

| $H_t$ | **Stochastic properties** | **Agents' beliefs** | **Market pattern** |
|---|---|---|---|
| $> \frac{1}{2}$ | Persistence<br>Smooth paths<br>$\langle X \rangle_{2,t} = 0$ | New information confirm outstanding positions | "*Low*" volatility - Momentum<br>Positive inefficiency (**PI**)<br>Overconfidence/Underreaction |
| $= \frac{1}{2}$ | Independence<br>Martingale<br>$\langle X \rangle_{2,t} = 2$ | Information fully incorporated by prices | "*Normal*" volatility<br>Sideways market<br>Efficiency (**E**) |
| $< \frac{1}{2}$ | Mean-reversion<br>Rough paths<br>$\langle X \rangle_{2,t} = \infty$ | New information disrupt outstanding positions | "*High*" volatility - Reversals<br>Negative inefficiency (**NI**)<br>Overreaction |

Figure: Financial interpretation of $H_t$.

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# Local Hurst Exponent - 1/5

The equation (6) suggests that we can obtain the estimation of the local $H_t$ by using a modification of variation statistics already introduced to estimate the parameter $H$ of a fBm.

Discretizing a MPRE as a $(X_i, i \in [\![1, n]\!])$ with time $t = \dfrac{i-1}{n-1}$, we can introduce a window of length $\nu << n$ where regularity is assumed constant and write:

$$M_t^{(2)} = \frac{1}{\nu} \sum_{j=i-\frac{\nu-1}{2}}^{i+\frac{\nu-1}{2}} |X_{j+1} - X_j|^2 \sim \mathcal{N}\left(0, C^2 \left(\frac{1}{n-1}\right)^{2H_t}\right) \tag{11}$$

with $i = \left[\!\left[1 + \frac{\nu-1}{2}, n - \frac{\nu-1}{2}\right]\!\right]$.

If a r.v. $Y \sim \mathcal{N}(0, \sigma^2)$, then $\mathbb{E}(|Y|^k) = \dfrac{2^{k/2}\Gamma((k+1)/2)}{\Gamma(1/2)}\sigma^k$. For this reason the quantity

$$M_t^{(k)} = \frac{1}{\nu} \sum_{j=i-\frac{\nu-1}{2}}^{i+\frac{\nu-1}{2}} |X_{j+1} - X_j|^k, \quad k > 0$$

satisfies

$$\mathbb{E}\left(M_t^{(k)}\right) = \frac{2^{k/2}\Gamma((k+1)/2)}{\Gamma(1/2)} C^k \left(\frac{1}{n-1}\right)^{kH_t}.$$

In particular for $k = 2$ we have

$$\mathbb{E}\left(M_t^{(2)}\right) = C^2 \left(\frac{1}{n-1}\right)^{2H_t}.$$

By halving the points in the window of length $\nu$ we obtain

$$M_t^{'(2)} = \frac{2}{\nu} \sum_{j=i-\frac{\nu-1}{2}}^{i+\frac{\nu-1}{2}} |X_{2j+1} - X_{2j-1}|^2$$

and we observe that, as $n \to \infty$

$$\lim_{\nu\to\infty} P\left(\frac{M_t^{'(2)}}{M_t^{(2)}} = 2^{2H_t}\right) = 1.$$

From these expressions one has the estimator

**Unbiased estimator with a low rate of convergence**

$$\hat{H}_t^{2,\nu,n} = \frac{1}{2}\log_2 \frac{M_t^{'(2)}}{M_t^{(2)}}. \tag{12}$$

The estimator in (12) is used to correct the

**Biased estimator - Bianchi (2005)**

$$\hat{H}_t^{\nu,n,C^*} = -\frac{\log\left(\frac{1}{\nu}\sum_{j=i-\frac{\nu-1}{2}}^{i+\frac{\nu-1}{2}} |X_{j+1} - X_j|^2\right)}{2\log(n-1)} + \frac{\log C^*}{\log(n-1)} \tag{13}$$

with $C^*$ arbitrarily chosen.

More precisely, the following Proposition holds

**Proposition**

Let $\hat{H}_t^{2,\nu,n}$ and $\hat{H}_t^{\nu,n,C^*}$ as in (12) and (13), respectively. Then the estimator

$$\hat{H}_t^{(\nu)} = \hat{H}_t^{\nu,n,C^*} - h \tag{14}$$

with

$$h := \frac{\log(C^*/C)}{\log(n-1)} = \hat{H}_t^{\nu,n,C^*} - \hat{H}_t^{2,\nu,n} + \xi_t,$$

is unbiased, does not depend on $C$.

# Data analysis - 1/4

To show the bias that can affect the estimate of a global $H$, we implement the following steps:

- Given the model (9), we generate $H_t \in [H_{\min}, H_{\max}] \subset (0,1)$ as a fOU with a prescribed $H$;
- Simulate a MPRE driven by the previous $H_t$;
- Set $\nu$ and estimate with the LHE the $\hat{H}_t^{(\nu)}$ from the MPRE;
- Estimate the global $\hat{H}_{\hat{H}_t^{(\nu)}}$ from the sequence $\{(t, \hat{H}_t^{(\nu)})\}$ using the Absolute Moment method (AM), the Aggregated Variance (AV) and the Higuchi's method (HH).

$$H \xrightarrow{fOU} H_t \rightarrow B_t^{H_t,C} \xrightarrow{LHE} \hat{H}_t^{(\nu)} \rightarrow \hat{H}_{\hat{H}_t^{(\nu)}}$$

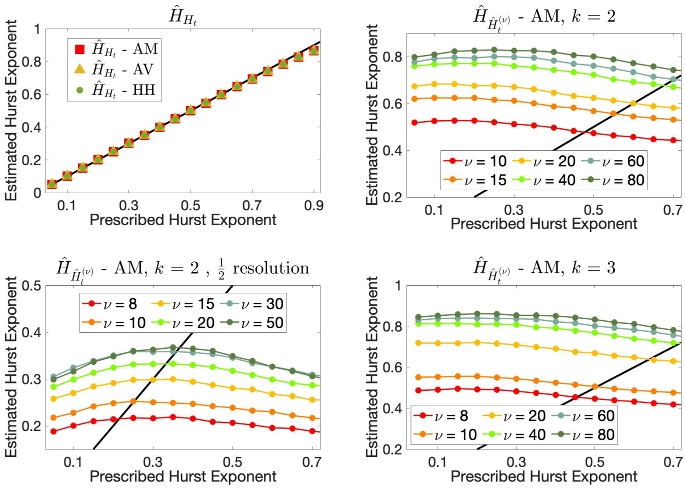


Figure: To test the stability of our results we estimate the global exponent with different order of variation $k$ and with a different the resolution of data.

# Data Analysis - 3/4

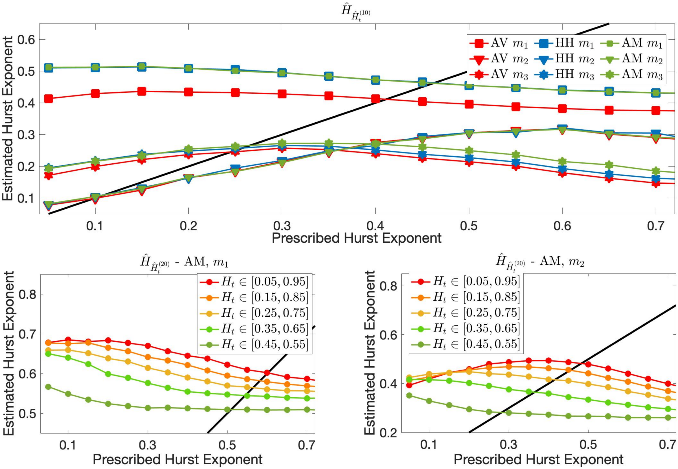


Figure: We test the estimators with different block set $m_1$, $m_2$, $m_3$ and for different range of $H_t$.

# Related Literature

- **Rough volatility**
  - Cont and Das (2022); Fukasawa et al. (2019); Fukasawa (2021); Gatheral et al. (2018); Takaishi (2020).
- **Multifractional processes**
  - Ayache and Bouly (2022); Ayache and Taqqu (2005); Ayache and Véhel (2004); Ayache et al. (2018); Bianchi (2005); Bianchi et al. (2015); Brandi and Di Matteo (2022); Loboda et al. (2021); Péltier and Véhel (1995).
- **Local Hurst exponent**
  - Benassi et al. (2000); Bianchi (2005); Bianchi et al. (2013); Couerjolly (2001); Istas and Lang (1997); Kent and Wood (1997); Pianese et al. (2018).
- **Global Hurst exponent**
  - Beran (1994); Garcin (2019); Higuchi (1988); Taqqu et al. (1995).

**Thank you very much for your attention!**