Multifractional processes and volatility

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Outline

Introduction: historical overview

Fractional Brownian motion

The best approach to introduce the multifractional processes is through a very well-known special case: the fractional Brownian motion (fBm).

Main References

- Mandelbrot B., *The variation of certain speculative prices*, J. Bus., XXXVI, 1963, 392-417
- Mandelbrot B., Taylor H., *On the distribution of stock price differences*, Oper. Res., 15, 1967, 1057-1062.
- Mandelbrot B., Van Ness JW. *Fractional Brownian, motion fractional noise and application*. SIAM Rev., 10, 1968,422–37

Remark

Reams and reams have been written about the incapability of the traditional Gaussian tools to capture the well-known stylized facts displayed by financial time series and covered in a celebrated work by Cont

Fractional Brownian Motion (fBm)

- The fractional Brownian motion (fBm) of Hurst or Hölder parameter $\boldsymbol{H} \in (\mathbf{0}, \mathbf{1})$ admits the following moving average representation:

$$B_H(t) = KV_H \int_{\mathbb{R}} (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} dW(s) \tag{1}$$

where $x_+ = max(x, 0)$, $V_H = \frac{\tau(2H+1)sin(\pi H)^{\frac{1}{2}}}{\tau(H+\frac{1}{2})}$ is a normalizing factor, $K^2 = VarB_H(1)$ while dW denotes the Brownian measure

- fBm increments are a zero-mean Gaussian, stationary, selfsimilar sequence with an autocovariance function such that:
  - if $\mathbf{0} < \boldsymbol{H} < \frac{\mathbf{1}}{\mathbf{2}}$ the motion displays antipersistence (negative increments tend to be followed by positive increments and vice versa);
  - if $\boldsymbol{H} = \frac{\mathbf{1}}{\mathbf{2}}$ fBm reduces to the ordinary Brownian motion.
  - if $\frac{\mathbf{1}}{\mathbf{2}} < \boldsymbol{H} < \mathbf{1}$ fBm shows persistence (negative/positive increments tend to be followed by negative/positive increments)

Fractional Brownian Motion (fBm)

- Let $X(t, \omega)$ be a stochastic process with a.s. continuous and not differentiable trajectories over the real line $\mathbb{R}$
- the local Hölder regularity of the path $t \mapsto X(t, \omega)$ w.r.t. some fixed point t can be measured by its **pointwise Hölder exponent** at t

$$\tilde{\alpha}_X(t, \omega) = \sup\left\{\tilde{\alpha} \geq 0 : \limsup_{h \to 0} \frac{|X(t+h, \omega) - X(t, \omega)|}{|h|^{\tilde{\alpha}}} = 0\right\} \quad (2)$$

- the smoother the path at time t the larger $\tilde{\alpha}$, the rougher the path the lower $\tilde{\alpha}$
- when X_t is a fractional Brownian motion of parameter H, $\tilde{\alpha}_X = H$ almost surely.

Fractional Brownian Motion (fBm)

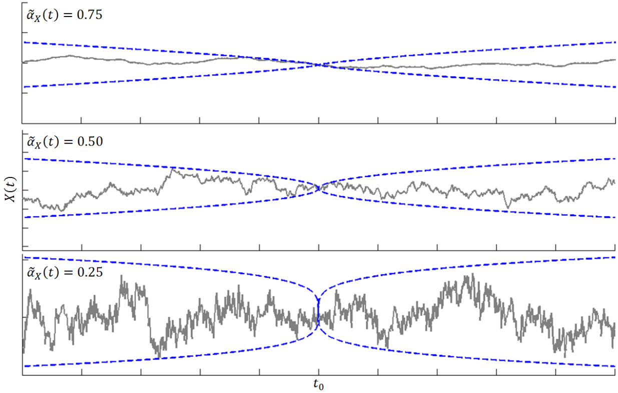

Figure: **Geometrical interpretation.** X has exponent $\tilde{\alpha}_X$ at t_0 if for any positive ϵ, a neighborhood $I(t_0)$ exists such that, for $t \in I(t_0)$, the graph of X is almost surely included in the envelop defined by $t \mapsto X(t_0) - C|t - t_0|^{\alpha-\epsilon}$ and $t \mapsto X(t_0) + C|t - t_0|^{\alpha+\epsilon}$.

Multifractional Brownian motion (mBm)

- Péltier and Lévy Véhel (1995) introduced the Multifractional Brownian motion (mBm) as generalization the fBm, by allowing to the Hurst parameter to change over time, becoming in this way an Hölderian function

$$\boldsymbol{H} : (\mathbf{0}, \infty) \to (\mathbf{0}, \mathbf{1})$$

- Unlike fBm's, the increments of mBm are no longer stationary nor self-similar.
- denoted by $Z(t, au) = B_{H(t+au)}(t + au) - B_{H(t)}(t)$ the increment process of the mBm at time t, and lag au, and by $X_{H(t)}$ a fBm with parameter $H(t)$, then (**local asymptotical self-similarity**)

$$\lim_{a \to 0^+} a^{-H(t)} Z(t, au) \stackrel{d}{=} X_{H(t)}(u), \quad u \in \mathbb{R} \tag{3}$$

- more flexibility with respect to fBm in order to describe those dynamics whose regularity changes over time

Table: Financial interpretation of H_t

H_t	Stochastic pattern	Investors' belief	Market pattern
$> \frac{1}{2}$	**Persistent** Smooth paths $\mathrm{QV}^{(*)} = 0$	New information confirmative of outstanding positions	**Low volatility** - Momentum Positive inefficiency Underreaction
$= \frac{1}{2}$	Independence **Martingale** $0 \neq \mathrm{QV} < \infty$	Information fully incorporated by prices	Normal volatility **Efficiency** Sideways market
$< \frac{1}{2}$	**Mean-reversion** Rough paths $\mathrm{QV} = \infty$	New information disruptive of outstanding positions	**High volatility** - Reversals Negative inefficiency Overreaction

$^{(*)}$ Quadratic Variation

Multifractional Brownian motion (mBm)

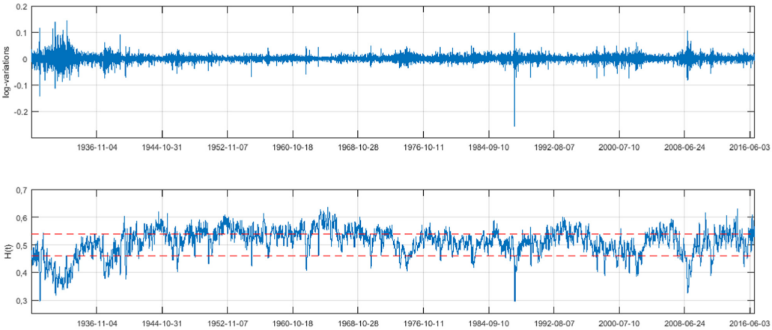

Figure: DJIA: (Top) Log-variations, (Bottom) Estimated Hölderian function

Multifractional Process with Random Exponent (MPRE)

- Ayache and Taqqu (2005) defined the Multifractional Process with Random Exponent (MPRE). as generalization the mBm replacing the deterministic function $H(t)$ by a stochastic process $\boldsymbol{S(t,\omega)}$
- The MPRE is the composition of two functions, the first one f_1 which provides the random parameter and the second f_2 that rules the resulting process.

$$\begin{matrix} H \\ (fBm) \end{matrix} \rightarrow \begin{matrix} H(t) \\ (mBm) \end{matrix} \rightarrow \begin{matrix} S(t,\omega) \\ (MPRE) \end{matrix}$$

Current lines of research

1. Stochastic modeling of the random exponent $\boldsymbol{S(t,\omega)}$ of MPRE
   - Square Root Process
   - Fractional Brownian Bridge
   - Fractional Ornstein-Uhlenbeck process

2. Stylized fact: volume-volatility relationship
   - Mixture of Distributions Hypothesis (MDH - Clark 1973)
   - Sequential Arrival of Information Hypothesis (SAIH - Copeland 1976)

Square root process and Fractional Brownian Bridge

a) The **square root process** is governed by the following stochastic differential equation

$$dS(t) = -\theta(S(t) - \mu)dt + \sigma\sqrt{S(t)}dW(t) \tag{4}$$

where

- μ: steady value (long-run mean or equilibrium level);
- θ: mean-reversion speed.

b) Denoted by B_t^H a fractional Brownian motion of parameter H, then the corresponding **fractional Brownian bridge** reads as

$$(B_t^H)^S := B_t^H - \frac{1}{2}\left(B_T^H - a\right)\left[1 + \left(\frac{t}{T}\right)^{2H} - \left(1 - \frac{t}{T}\right)^{2H}\right] \tag{5}$$

fractional Brownian bridge

- Using relation (5), surrogate paths of fBb of length T (representing the simulated dynamics of S_t) as follows:

$$S_t^{\text{sim}} = \left\{ B_t^H - \frac{1}{2}\left(B_T^H - \mathbb{A}\right)\left[1 + \left(\frac{t}{T}\right)^{2H} - \left(1 - \frac{t}{T}\right)^{2H}\right]\right\}\sigma_{\widetilde{H}_{\nu,n}} + \frac{1}{2}$$

- the r.v. $\mathbb{A}$ forces the terminal values of all simulated paths of fBb to distribute as the standardized $\widetilde{H}_{\nu,n}$,
- the scaling parameter $\sigma_{\widetilde{H}_{\nu,n}}$ will ensure that the intermediate values of the surrogated fBb deviate from 0 as the observed ones
- the quantity 1/2 ensures that the surrogates H_t will fluctuate around the equilibrium value

fractional Brownian bridge

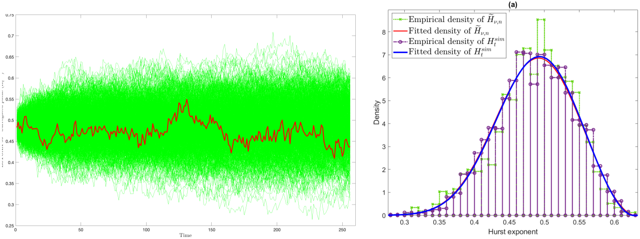

Figure: ESTOXX50: January 2, 2000 - December 31, 2021 (left panel) 1.000 sample paths (in red one of the paths); (right panel) fits of simulations.

fractional Ornstein–Uhlenbeck process

- The fractional Ornstein–Uhlenbeck process X_t is defined as the stationary solution of the stochastic differential equation

$$dX_t = -\alpha(X_t - m)dt + \nu dB_t^H, \tag{6}$$

where $m \in \mathbb{R}$, ν and α are positive parameters. As for usual Ornstein–Uhlenbeck processes, there is an explicit form for the solution which is given by

$$X_t = m + \nu \int_{-\infty}^{t} e^{-\alpha(t-s)} dB_s^H, \tag{7}$$

where the stochastic integral with respect to fBm is a pathwise Riemann–Stieltjes integral

The volume-volatility relationship

- In a celebrated paper, Cont (2001) introduces the so-called stylized facts displayed by financial time series; among them a remarkable role is played by the volume-volatility relationship.
- The **MDH** posits that both volume and volatility are driven by the same underlying information flow; as a consequence, they change **simultaneously** as soon as information is processed by market participants $\Rightarrow$ **positive correlation**
- The **SAIH** posits that information dissemination occurs **sequentially** to investors, causing a series of intermediate equilibrium prices, and thus leading to a final informational equilibrium price when all the investors are informed $\Rightarrow$ **lead-lag relation**

The volume-volatility relationship

- The findings strongly depend on:
  a) the volatility measure and the calculation methodology
  b) the time period and the expanding-contracting market
  c) the time frames (intraday, daily, weekly . . .),

- Bollerslev (2018) et al. observe that
  "..the MDH provides a possible statistical explanation for the positive volume-volatility relationship based on the idea of a common news arrival process driving both the magnitude of returns and trading volume. The MDH, however, remains silent about the underlying economic mechanisms that link the actual trades and price adjustements to the news".

- the key explanatory variable that impacts all three aspects (a)-(c): **market efficiency**.

THANK YOU!