Optimal reinsurance problems for jump-clusters models

–Joint work with M. Brachetta, G. Callegaro and C. Sgarra–

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Contents

Preliminary considerations

- Optimal reinsurance-investment problem is one of the core research problems in actuarial science. Purchasing reinsurance can protect insurers against adverse claim experience.
- There exists a large literature on this topic, under different criteria (e.g., minimizing ruin probability or maximizing expected utility). See for instance, among others, [Schmidli 2007], [Liang et al. IME 2014], [Zhang et al. IME 2009], [Zhu et al. IME 2015].
- Most of the literature is based on the classical Cramér-Lundberg model or its diffusion approximation.

- Classical models assume constant claims arrival intensity .
- This assumption is often far from realistic. For instance:
  - Car's insurance claims may be influenced by weather conditions;
  - Claims associated with natural catastrophes are in general affected by environmental stochastic factors;
  - Claims induced by terrorist attacks are influenced by social and political conditions;
- In many cases, these stochastic factors are not directly observable by insurance companies. This leads to discuss the problem under partial information.

Main ingredients

- **JUMP CLUSTERING:** in catastrophic situations the jumps in the claims arrival process can exhibit clustering feature. We combine Cox with shot-noise intensity and Hawkes processes (with exponential kernel) and we get a shot-noise self-exciting counting process
- **PARTIAL INFORMATION:** insurer has partial information about claims arrival intensity.

Related literature

Partial Information:

- Liang, Z., Bayraktar, E. (2014): Optimal reinsurance and investment with unobservable claim size and intensity. *Insurance Math. Econom.* 55.
- Brachetta, M., Ceci, C. (2020): A BSDE-based approach for the optimal reinsurance problem under partial information, *Insurance Math. Econom.* 95

Contagion model:

- Dassios A. , Zhao, H. (2011): A dynamic contagion process, *Adv. Appl. Prob.* 43.
- Cao Y., Landriault D., Li, B. (2020): Optimal reinsurance-investment strategy for a dynamic contagion claim model. *Insurance Math. Econom.* 93.

The Mathematical Model

On $(\Omega, \mathscr{F}, \mathbf{P}; \mathbb{F})$ with $T > 0$ the maturity of a reinsurance contract, introduce the cumulative claim process $C = \{C_t,\ t \in [0, T]\}$:

$$C_t = \sum_{j=1}^{N_t^{(1)}} \underbrace{Z_j^{(1)}}_{claims\ size} \quad , \quad t \in [0, T]$$

where the claims arrival process $N^{(1)}$ is a point process with intensity:

$$\lambda_t = \beta + (\lambda_0 - \beta) e^{-\alpha t} + \underbrace{\sum_{j=1}^{N_t^{(1)}} e^{-\alpha(t - T_j^{(1)})} \ell(\underbrace{Z_j^{(1)}}_{Int-exc.jump})}_{CLUSTERING} + \underbrace{\sum_{j=1}^{N_t^{(2)}} e^{-\alpha(t - T_j^{(2)})} \underbrace{Z_j^{(2)}}_{Ext-exc.jump}}_{CLUSTERING}$$

Assumption

$N^{(2)}$ Poisson process with intensity $\rho > 0$; $\{Z_n^{(1)}\}_{n \geq 1}$ ($\{Z_n^{(2)}\}_{n \geq 1}$) i.i.d. $\mathbb{R}^+$-valued rv with distribution function $F^{(1)}$ ($F^{(2)}$). $N^{(2)}$, $\{Z_n^{(1)}\}_{n \geq 1}$ and $\{Z_n^{(2)}\}_{n \geq 1}$ are independent.

Model Construction

The key idea is based on equivalent change probability measure on $(\Omega, \mathscr{F}; \mathbb{F})$.

Under **Q**:

- $N^{(1)}$ and $N^{(2)}$ are Poisson processes with intensity 1 and $\rho > 0$, respectively;
- the integer valued random measures $m^{(i)}(\mathrm{d}t, \mathrm{d}z)$, $i = 1, 2$

$$m^{(i)}(\mathrm{d}t, \mathrm{d}z) = \sum_{n \geq 1} \delta_{(T_n^{(i)}, Z_n^{(i)})}(\mathrm{d}t, \mathrm{d}z) \mathbb{1}_{\{T_n^{(i)} < \infty\}}.$$

  Under **Q**: $m^{(i)}(\mathrm{d}t, \mathrm{d}z)$, $i = 1, 2$, are independent Poisson measures with compensator measures given respectively by

$$\nu^{(1),\mathbf{Q}}(\mathrm{d}t, \mathrm{d}z) = F^{(1)}(\mathrm{d}z)\mathrm{d}t, \quad \nu^{(2),\mathbf{Q}}(\mathrm{d}t, \mathrm{d}z) = \rho F^{(2)}(\mathrm{d}z)\mathrm{d}t.$$

- By Girsanov Theorem under P the $(\mathbf{P}, \mathbb{F})$-predictable projections measures of the random measure $m^{(i)}(\mathrm{d}t, \mathrm{d}z)$, $i = 1, 2$ are given by:

$$\nu^{(1)}(\mathrm{d}t, \mathrm{d}z) = \lambda_{t^-} F^{(1)}(\mathrm{d}z)\mathrm{d}t, \quad \nu^{(2)}(\mathrm{d}t, \mathrm{d}z) = \rho F^{(2)}(\mathrm{d}z)\mathrm{d}t. \qquad (1)$$

  In particular, $N^{(1)}$ is a point process with $(\mathbf{P}, \mathbb{F})$-predictable intensity $\{\lambda_{s^-}\}_{s \in [0,T]}$.

Reinsurance Contract

The insurer selects a reinsurance strategy $\{u_t\}_{t\in[0,T]}$, so that the aggregate losses covered by the insurer are

$$C_t^u = \sum_{j=1}^{N_t^{(1)}} \Phi(Z_j^{(1)}, u_{T_j^{(1)}}) = \int_0^t \int_0^{+\infty} \Phi(z, u_s) m^{(1)}(\mathrm{d}s, \mathrm{d}z), \quad t \in [0, T],$$

(the remaining $C_t - C_t^u$ will be undertaken by the reinsurer). We assume:

- The retention function $\Phi(z, u)$ continuous in $u \in U$;
- $U \subseteq \overline{\mathbb{R}}^n$, with $\overline{\mathbb{R}}$ denoting the compactification of $\mathbb{R}$;
- There exists at least two points u_N and $u_M \in U$ such that

$$0 \leq \Phi(z, u_M) \leq \Phi(z, u) \leq \Phi(z, u_N) = z, \quad \forall u \in U$$

(u_M=maximal reinsurance, u_N=null reinsurance).

Different type of contracts

a) Proportional reinsurance: the insurer transfers a percentage $1-u$ of any future loss to the reinsurer, so $U=[0,1]$ and $\Phi(z,u)=uz$.

b) Excess-of-loss: the reinsurer covers all the losses exceeding a threshold u, hence $U=[0,+\infty]$ and $\Phi(z,u)=u\wedge z$.

c) Limited excess of loss reinsurance: the reinsurer covers the losses exceeding a threshold u_1, up to a maximum level $u_2>u_1$, so that the maximum loss is limited to (u_2-u_1) on the reinsurer's side. In this case: $\Phi(z,u)=z-(z-u_1)^+ + (z-u_2)^+$, so that $U=\{(u_1,u_2): u_1\geq 0, u_2\in[u_1,+\infty]\}$ and $u=(u_1,u_2)$. Here $u_M=(u_{M,1},u_{M,2})=(0,+\infty)$ and u_N can be any point on the line $u_1=u_2$.

d) Limited excess of loss with fixed reinsurance coverage: $u_2=u_1+\beta$, $\beta>0$. Here $U=[0,+\infty]$, $u_N=+\infty$ and $u_M=0$ corresponds to the maximum reinsurance coverage β.

The surplus and the reinsurance premium

Under $\{u_t\}_{t\in[0,T]}$, the surplus process R^u of the primary insurer follows:

$$dR_t^u = \left(c_t - q_t^u\right) dt - dC_t^u, \quad R_0^u = R_0 \in \mathbb{R}^+$$

with $\mathbb{H}-$predictable processes

- c_t insurance premium rate;
- the reinsurance premium rate, $q_t^u(\omega) = q(t,\omega,u)$ is such that
  - $q(t,\omega,u)$ and $\frac{\partial q(t,\omega,u)}{\partial u}$ continuous in $u \in U$,
  - $q(t,\omega,u_N) = 0$ null protection is not expensive,
  - $q(t,\omega,u_M) > q(t,\omega,u)$, the maximum reinsurance is the most expensive.

Assumption

$$\mathbb{E}\left[\int_0^T q_t^{u_M} \mathrm{d}t\right] < \infty, \quad \mathbb{E}\left[\int_0^T c_t \mathrm{d}t\right] < \infty$$

The wealth and the problem to solve

The insurance company invests its surplus in a risk-free asset with interest rate $r > 0$, so that the wealth is $X_0^u = R_0 \in \mathbb{R}^+$

$$dX_t^u = dR_t^u + rX_t^u\,dt = \left(c_t - q_t^u\right)dt - \int_0^{+\infty} \Phi(z, u_t)\, m^{(1)}(dt, dz) + rX_t^u\,dt$$

Under Full Information $m^{(1)}(dt, dz)$ has $(\mathbf{P}, \mathbb{F})$-compensator measure $\lambda_{t^-} F^{(1)}(\mathrm{d}z)\mathrm{d}t$;
Under Partial Information: $m^{(1)}(dt, dz)$ has $(\mathbf{P}, \mathbb{H})$-compensator measure $\pi_{t^-}(\lambda) F^{(1)}(\mathrm{d}z)\mathrm{d}t)$, where $\mathbb{H} = \mathbb{F}^C$ and $\pi_t(\lambda) = \mathbb{E}[\lambda_t | \mathscr{H}_t]$.

The company aims at solving (with $\eta > 0$ the insurer's risk aversion)

$$\sup_{u \in \mathscr{U}} \mathbb{E}\left[1 - e^{-\eta X_T^u}\right] = 1 - \inf_{u \in \mathscr{U}} \mathbb{E}\left[e^{-\eta X_T^u}\right]$$

The Admissible strategies are all the U-valued, $\mathbb{F}$-(or $\mathbb{H}$)-predictable processes.

HJB-approach under full information

- Assuming $c_t = c(t, \lambda_t)$ and $q^u_t(t, u_t, \lambda_t)$;
- (X^u_t, λ_t) is a Markov process;
- Value function

$$V(t, x, \lambda) = \inf_{u \in \mathcal{U}} \mathbb{E}_{t,x,\lambda}\left[e^{-\eta X^u_T}\right], \quad (t, x, \lambda) \in [0, T) \times \mathbb{R} \times (0, +\infty),$$

  where the notation $\mathbb{E}_{t,x,\lambda}[\cdot]$ stands for the conditional expectation given $X^u_t = x$ and $\lambda_t = \lambda$.
- We can prove that $V(t, x, \lambda) = e^{-\eta x e^{r(T-t)}} \varphi(t, \lambda)$ and if φ is sufficiently smooth it solves the Hamilton-Jacobi-Bellman (HJB) equation.

The Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{\partial \varphi}{\partial t}(t,\lambda) + \alpha(\beta - \lambda)\frac{\partial \varphi}{\partial \lambda}(t,\lambda) + \int_0^{+\infty} [\varphi(t,\lambda+z) - \varphi(t,\lambda)]\, \rho F^{(2)}(\mathrm{d}z) \\ - \eta e^{r(T-t)} \varphi(t,\lambda) c(\lambda) + \inf_{u\in[0,1]} \Psi^u(t,\lambda) = 0, \tag{2}$$

with final condition $\varphi(T,\lambda) = 1, \quad \lambda \in (0,+\infty)$, where the function Ψ^u is given by

$$\Psi^u(t,\lambda) = \eta e^{r(T-t)} \varphi(t,\lambda) q(\lambda,u) + \int_0^{+\infty} \left[e^{\eta \Phi(z,u) e^{r(T-t)}} \varphi(t,\lambda+\ell(z)) - \varphi(t,\lambda) \right] \lambda F^{(1)}(\mathrm{d}z).$$

Difficulties:

- Regularity of the value function;
- Verification approach requires to prove existence and uniqueness of the solution Eq.(2) (partial integro-differential equation with an embedded optimization);
- Two alternative approaches: direct computations or a BSDEs-approach (in collaboration with Alessandra Cretarola, Universitá di Perugia).

BSDE-approach

We define, for $\mathcal{U}(t,u) = \left\{ \bar{u} \in \mathcal{U} : \bar{u}_s = u_s \text{ a.s., } s \le t \le T \right\}$, the Snell envelope

$$W_t^u = \operatorname*{ess\,inf}_{\bar{u} \in \mathcal{U}(t,u)} \mathbb{E}\left[e^{-\eta X_T^{\bar{u}}} \mid \mathcal{H}_t \right],$$

so that if $\widehat{X}_t^u := e^{-rt} X_t^u$ is the discounted wealth, then $W_t^u = e^{-\eta \widehat{X}_t^u e^{rT}} V_t$, where V is the value process: $V_t = \operatorname{ess\,inf}_{\bar{u} \in \mathcal{U}_t} \mathbb{E}\left[e^{-\eta e^{rT} (\widehat{X}_T^{\bar{u}} - \widehat{X}_t^{\bar{u}})} \mid \mathcal{H}_t \right]$ Moreover, $V_t = e^{\eta \widehat{X}_t^{u_N} e^{rT}} W_t^N$.

Proposition (Bellman's Optimality Principle)

i) $\{W_t^u, t \in [0,T]\}$ *in a* $(\mathbf{P}, \mathbb{H})$*-submartingale* $\forall\, u \in \mathcal{U}$;

ii) $\{W_t^{u^*}, t \in [0,T]\}$ *in a* $(\mathbf{P}, \mathbb{H})$*-martingale if and only if* $u^* \in \mathcal{U}$ *is an optimal control.*

Theorem (Main Result)

$(W^N, \Theta^{W^N}) \in \mathscr{L}^2 \times \widehat{\mathscr{L}^2}$ *is the unique solution the following BSDE*

$$W_t^N = \xi - \int_t^T \int_0^{+\infty} \Theta_s^{W^N}(z)\ \widetilde{m}^{(1)}(ds, dz) - \int_t^T \operatorname*{ess\,sup}_{u \in \mathscr{U}} \widetilde{f}(s, W_s^N, \Theta_s^{W^N}(\cdot), u_s)\, ds, \tag{3}$$

with terminal condition $\xi = e^{-\eta X_T^N}$*, where*

$$\widetilde{f}(t, W_t^N, \Theta_t^{W^N}(\cdot), u_t) = -W_{t-}^N \eta e^{r(T-t)} q_t^u - \int_0^{+\infty} [W_{t-}^N + \Theta_t^{W^N}(z)]\left[e^{-\eta e^{r(T-t)}(z - \Phi(z, u_t))} - 1\right] \pi_{t\_}(\lambda) F^{(1)}(dz). \tag{4}$$

Moreover, the process $u^* \in \mathscr{U}$ *which satisfies*

$$\widetilde{f}(t, W_t^N, \Theta_t^{W^N}(\cdot), u_t^*) = \operatorname*{ess\,sup}_{u \in \mathscr{U}} \widetilde{f}(t, W_t^N, \Theta_t^{W^N}(\cdot), u_t) \qquad \forall t \in [0, T] \tag{5}$$

is an optimal control.

Proof.

It follows directly by an existence result of a solution to the BSDE (3) and a verification result, which imply that any solution to the BSDE (3) coincides with the process (W^N, Θ^{W^N}). □

Theorem (Existence result)

There exists a unique solution to the BSDE (3).

Proof. We prove that the driver satisfies a stochastic Lipschitz condition. We adapt to our framework Theorem 3.5 in Papapantoleon, Possamai and Saplaouras (EJP, 2018).

Theorem (Verification Theorem)

Let $(Y, \Theta^Y) \in \mathcal{L}^2 \times \widehat{\mathcal{L}}^2$ *be a solution to the BSDE* (3) *and let* $u^* \in \mathcal{U}$ *be the maximizer of* $\tilde{f}(t, Y_t, \Theta^Y_t(\cdot), u_t)$. *Then* $Y = W^N$ *and*

$$V_t = e^{\eta \bar{X}^{u_N}_t} e^{rT} Y_t \qquad \forall t \in [0, T],$$

and u^* *is an optimal control.*

Proportional reinsurance, $\Phi(z,u) = zu$

- Expected Value Principle: $q^u_t = (1+\theta_R)\mathbb{E}[Z^{(1)}]\pi_{t^-}(\lambda)(1-u_t)$
- The optimal control u^* is obtained "explicitly" and

$$u^*_t(\omega) = \begin{cases} 0 & \text{if } \theta_R < \theta^F_t(\omega) \\ 1 & \text{if } \theta_R > \theta^N_t(\omega) \\ \bar{u}(t, W^N_{t^-}(\omega), \Theta^{W^N}_t(\cdot)(\omega)) & \text{otherwise,} \end{cases}$$

The stochastic thresholds are:

$$\theta^F_t = \frac{1}{\mathbb{E}[Z^{(1)}]}\int_0^\infty \frac{W^N_{t^-} + \Theta^{W^N}_t(z)}{W^N_{t^-}} z e^{-\eta e^{r(T-t)z}} F^{(1)}(dz) - 1, \quad \theta^N_t = \frac{1}{\mathbb{E}[Z^{(1)}]}\int_0^\infty \frac{W^N_{t^-} + \Theta^{W^N}_t(z)}{W^N_{t^-}} z F^{(1)}(dz) - 1.$$

and $\bar{u}(t, W^N_{t^-}, \Theta^{W^N}_t(\cdot)) \in (0,1)$ solves the following equation:

$$(1+\theta_R)\mathbb{E}[Z^{(1)}] = \int_0^{+\infty} \frac{W^N_{t^-} + \Theta^{W^N}_t(z)}{W^N_{t^-}} z e^{-\eta e^{r(T-t)z(1-u)}} F^{(1)}(dz).$$

Limited Excess-of-Loss Reinsurance with fixed maximum reinsurance coverage $\beta > 0$

- According to the expected value principle

$$q_t^u = (1+\theta_R)\pi_{t^-}(\lambda)\int_{u_t}^{u_t+\beta}(1-F^{(1)}(z))dz.$$

- The optimal control u^* is given by

$$u_t^*(\omega) = \begin{cases} 0 & \text{if } \theta_R < \theta_t^L(\omega) \\ \bar{u}(t, W_{t^-}^N(\omega), \Theta_t^{W^N}(\cdot)(\omega)) & \text{otherwise,} \end{cases}$$

where

$$\theta_t^L = \frac{1}{F^{(1)}(\beta)}\int_0^\beta \frac{W_{t^-}^N + \Theta_t^{W^N}(z)}{W_{t^-}^N} e^{-\eta e^{r(T-t)z}} F^{(1)}(dz) - 1.$$

and $\bar{u}(t, W_{t^-}^N, \Theta_t^{W^N}(\cdot)) \in (0, +\infty)$ solves the following equation:

$$(1+\theta_R)\left(F^{(1)}(u+\beta) - F^{(1)}(u)\right) = \int_u^{u+\beta} \frac{W_{t^-}^N + \Theta_t^{W^N}(z)}{W_{t^-}^N} e^{-\eta e^{r(T-t)(z-u)}} F^{(1)}(dz).$$

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