One-Dimensional Proof of the Gaussian Integral

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Statement of the problem

The well-known Gaussian Integral, that we denote by G, is

$$G = \int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\,.$$

Its most famous and simplest proof is the one involving a double integral on the whole plane $\mathbb{R}^2$, that is

$$\left(\int_0^{+\infty} e^{-x^2}\,dx\right)^2 = \int_0^{+\infty} e^{-x^2}\,dx \int_0^{+\infty} e^{-y^2}\,dy =$$

$$= \int_{x,y\geqslant 0} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{+\infty} e^{-r^2} r\,dr \int_0^{\pi/2} d\theta = \frac{\pi}{4}\,.$$

Statement of the problem

Since this proof involves a double integral, we present a proof involving functions and sequences of one variable, only, by starting from the convergent series

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = e^{x^2},$$

from which we get the inequality $e^{x^2} \geqslant 1 + x^2$, that we rewrite in the form with the inverse of both sides

$$e^{-x^2} \leqslant \frac{1}{1+x^2}. \tag{1}$$

Statement of the problem

In order to prove the inequality $e^{-x^2} \geqslant 1-x^2$ for $0 \leqslant x \leqslant 1$, we write Taylor's formula of e^{-x^2} in $x_0 = 0$ by expanding the polynomial up to the order 2, with $0 \leqslant x \leqslant 1$ such that the point c, whose existence is stated by Taylor's theorem, belongs to the interval $0 < c < x \leqslant 1$.

Since the third derivative of e^{-x^2} is $4x(3-2x^2)e^{-x^2}$ and is always positive on $[0,1]$, we get Taylor's formula

$$e^{-x^2} = 1 - x^2 + \frac{2c(3-2c^2)e^{-c^2}}{3}\, x^3,$$

from which the inequality

$$e^{-x^2} \geqslant 1 - x^2 \tag{2}$$

follows, for $0 \leqslant x \leqslant 1$.

Inequalities for the Gaussian Integral

By writing together the inequalities (1) and (2), we have

$$1 - x^2 \leqslant e^{-x^2} \leqslant \frac{1}{1 + x^2},$$

for $0 \leqslant x \leqslant 1$, from which we obtain the inequality between the integrals on $[0, 1]$ of the positive integer n-th powers

$$\int_0^1 \left(1 - x^2\right)^n dx \leqslant \int_0^1 e^{-nx^2} dx \leqslant \int_0^1 \frac{dx}{(1 + x^2)^n}\,, \qquad \text{(3a)}$$

that is

$$\int_0^1 \left(1 - x^2\right)^n dx \leqslant \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-x^2} dx \leqslant \int_0^1 \frac{dx}{(1 + x^2)^n}\,, \qquad \text{(3b)}$$

where for the second integral in (3a) we have changed $x\sqrt{n} = y$.

Inequalities for the Gaussian Integral

If we denote the first and third integral in (3b) as sequences

$$J_n = \int_0^1 \left(1 - x^2\right)^n dx \qquad \text{and} \qquad K_n = \int_0^1 \frac{dx}{(1+x^2)^n}\,, \tag{4a}$$

we can finally write the inequality

$$\sqrt{n}\, J_n \leqslant \int_0^{\sqrt{n}} e^{-x^2} dx \leqslant \sqrt{n}\, K_n\,, \tag{4b}$$

to which we apply the *sandwich theorem* that will give us

$$\lim_{n\to+\infty} \sqrt{n}\, J_n = \frac{\sqrt{\pi}}{2} \leqslant \int_0^{+\infty} e^{-x^2} dx \leqslant \frac{\sqrt{\pi}}{2} = \lim_{n\to+\infty} \sqrt{n}\, K_n\,.$$

Equality for Wallis product

In order to compute the two limits

$$\lim_{n\to+\infty} \sqrt{n}\, J_n \qquad \text{and} \qquad \lim_{n\to+\infty} \sqrt{n}\, K_n$$

of the first and third sequences in (4b), let us prove that the equality

$$\frac{16^n\,(n!)^4}{(2n)!\,(2n+1)!} = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \tag{5}$$

holds for all positive integer index n, where the product in the right-hand side of (5) is called *Wallis Product*.

Equality for Wallis product

If we define the sequence

$$a_n := \frac{16^n\,(n!)^4}{(2n)!\,(2n+1)!} - \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}\,,$$

it is straightforward to notice that the following relations

$$\begin{cases} a_1 = \dfrac{16^1\,(1!)^4}{(2!)\,(3!)} - \displaystyle\prod_{k=1}^{1} \frac{4k^2}{4k^2-1} = \frac{16}{12} - \frac{4}{3} = 0, \\[2ex] a_{n+1} = \dfrac{4\,(n+1)^2}{(2n+1)(2n+3)}\,a_n \end{cases}$$

hold, from which, by induction, we get $a_n = 0$, for all positive integer index n, and then the equality (5) is true for all positive integer n.

Limit of Wallis product

In order to compute the limit of *Wallis product*

$$\lim_{n\to+\infty}\prod_{k=1}^{n}\frac{4k^2}{4k^2-1},$$

we compute the limit of the *inverse Wallis product*

$$\lim_{n\to+\infty}\prod_{k=1}^{n}\frac{4k^2-1}{4k^2}=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{36}\right)\cdots\left(1-\frac{1}{4n^2}\right)=$$

$$=\lim_{n\to+\infty}\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right)\bigg|_{x=\pi/2}=$$

$$=\lim_{n\to+\infty}\left[\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\right]\left[\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\right]\cdot$$

$$\cdot\left[\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\right]\cdots\left[\left(1-\frac{x}{n\pi}\right)\left(1+\frac{x}{n\pi}\right)\right]\bigg|_{x=\pi/2}.$$

Limit of Wallis product

It is straightforward to recognize that the limit

$$\lim_{n\to+\infty}\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right)=$$

$$=\lim_{n\to+\infty}\left[\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\right]\left[\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\right]\cdot$$

$$\cdot\left[\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\right]\cdots\left[\left(1-\frac{x}{n\pi}\right)\left(1+\frac{x}{n\pi}\right)\right]$$

is the infinite factorization of the *Taylor's series* of an *even function*, that we denote by $f(x)$, such that

$$f(0)=1 \tag{6a}$$

and whose *zeros* are

$$x_k = k\pi,\ \text{ for all } k\in\mathbb{Z} \text{ with } k\neq 0. \tag{6b}$$

Limit of Wallis product

Since the only function satisfying the two properties (6) is

$$f(x) = \frac{\sin x}{x},$$

we get the limit of the *inverse Wallis product*

$$\lim_{n\to+\infty} \prod_{k=1}^{n} \frac{4k^2-1}{4k^2} = \lim_{n\to+\infty} \prod_{k=1}^{n} \left(1 - \frac{x^2}{k^2\pi^2}\right)\Bigg|_{x=\pi/2} =$$

$$= \frac{\sin x}{x}\Bigg|_{x=\pi/2} = \frac{2}{\pi}$$

and then the limit of *Wallis product*

$$\lim_{n\to+\infty} \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} = \frac{\pi}{2}. \tag{7}$$

We now determine the sequence J_n in (4a) by expanding the integral

$$J_n = \int_0^1 \left(1 - x^2\right)^n dx = \int_0^{\pi/2} (\cos t)^{2n+1}\, dt = 2nJ_{n-1} - 2nJ_n\,,$$

from which we get the "*Cauchy problem*"

$$\begin{cases} J_n = \dfrac{2n}{2n+1}\, J_{n-1}\,, \\[2ex] J_0 = \displaystyle\int_0^1 dx = 1, \end{cases}$$

which is a *homogeneous finite difference equation* of *first order*,

Sequence of the integrals J_n

whose solution is

$$J_n = \int_0^1 \left(1 - x^2\right)^n dx = \frac{4^n (n!)^2}{(2n+1)!}\,, \tag{8}$$

because by induction we have

$$J_1 = \frac{2}{3}\,, \quad J_2 = \frac{4}{5} \cdot \frac{2}{3}\,, \quad J_3 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}\,, \quad \dots\,, \quad J_n = \frac{(2n)!!}{(2n+1)!!}\,,$$

where

$$(2n)!! = (2n)(2n-2)(2n-4)\cdots(6)(4)(2) = 2^n \cdot n!$$

and

$$(2n+1)!! = (2n+1)(2n-1)(2n-3)\cdots(7)(5)(3) = \frac{(2n+1)!}{2^n \cdot n!}\,.$$

Limit of the product $\sqrt{n}\, J_n$

If we now divide by $2n+1$ both sides of (5), we obtain

$$\begin{aligned} \sqrt{n}\, J_n &= \sqrt{n}\left[\frac{(2n)!!}{(2n+1)!!}\right] = \sqrt{n}\left[\frac{4^n (n!)^2}{(2n+1)!}\right] = \\ &= \sqrt{\frac{n}{2n+1} \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}}\,, \end{aligned} \tag{9}$$

from which we get the limit of the sequence on the left in (4b)

$$\lim_{n \to +\infty} \sqrt{n}\, J_n = \lim_{n \to +\infty} \sqrt{\frac{n}{2n+1} \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}} = \frac{\sqrt{\pi}}{2}\,.$$

A series from the integrals J_n

If we now write (8) with the index k and divide its second and third side by a^k, with $a > 1$, the sum over k from 0 up to n actually reads

$$\sum_{k=0}^{n} \frac{(4/a)^k\,(k!)^2}{(2k+1)!} = \sum_{k=0}^{n} \int_0^1 \left(\frac{1-x^2}{a}\right)^k dx =$$

$$= \int_0^1 \sum_{k=0}^{n} \left(\frac{1-x^2}{a}\right)^k dx = \int_0^1 f_n(x)\,dx,$$

where

$$f_n(x) = \sum_{k=0}^{n} \left(\frac{1-x^2}{a}\right)^k = \frac{1-\left(\dfrac{1-x^2}{a}\right)^{n+1}}{1-\dfrac{1-x^2}{a}}.$$

A series from the integrals J_n

Since the sequence of functions $f_n(x)$ satisfies the inequality

$$|f_n(x)| \leqslant \frac{a}{a-1+x^2} \in L^1([0,1]),$$

we can apply *Lebesgue's dominated convergence theorem* to exchange limit and integral, from which we get

$$\sum_{k=0}^{+\infty} \frac{(4/a)^k \, (k!)^2}{(2k+1)!} = \lim_{n\to+\infty} \int_0^1 f_n(x)\, dx = \int_0^1 \lim_{n\to+\infty} f_n(x)\, dx =$$

$$= \int_0^1 \frac{a}{a-1+x^2}\, dx = \frac{a}{\sqrt{a-1}} \arctan\left(\frac{1}{\sqrt{a-1}}\right).$$

A series from the integrals J_n

By putting $a = 2$ in the first and the last term, the relation

$$\sum_{k=0}^{+\infty} \frac{2^k \, (k!)^2}{(2k+1)!} = \frac{\pi}{2} \tag{10}$$

follows.

Sequence of the integrals K_n

We now determine the sequence K_n in (4a) by expanding the integral

$$K_n = \int_0^1 \frac{dx}{(1+x^2)^n} = \int_0^{\pi/2} (\cos t)^{2n-2}\, dt =$$

$$= \frac{1}{2^{n-1}} + (2n-3)K_{n-1} - (2n-3)K_n\,,$$

from which we get the "*Cauchy problem*"

$$\begin{cases} K_n = \dfrac{2n-3}{2n-2}\, K_{n-1} + \dfrac{1}{2^n(n-1)}\,, \\[2ex] K_1 = \displaystyle\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}\,, \end{cases}$$

which is a *non-homogeneous finite difference equation* of *first order*, whose solution is

$$K_n = \frac{(2n-3)!!}{(2n-2)!!}\left[\frac{\pi}{4} + \sum_{k=0}^{n-2} \frac{(2k+2)!!}{(2k+1)!!\,2^{k+2}(k+1)}\right],$$

that, by virtue of the equality (9), can be rewritten in the form

$$K_n = \sqrt{\frac{1}{2n-1}\prod_{k=1}^{n-1}\frac{4k^2-1}{4k^2}}\;\left[\frac{\pi}{4} + \frac{1}{2}\sum_{k=0}^{n-2}\frac{2^k\,(k!)^2}{(2k+1)!}\right],$$

from which we get the limit

Limit of $\sqrt{n}\,K_n$

$$\lim_{n\to+\infty} \sqrt{n}\,K_n =$$

$$= \lim_{n\to+\infty} \sqrt{\frac{n}{2n-1}\prod_{k=1}^{n-1}\frac{4k^2-1}{4k^2}}\left[\frac{\pi}{4}+\frac{1}{2}\sum_{k=0}^{n-2}\frac{2^k\,(k!)^2}{(2k+1)!}\right] =$$

$$= \sqrt{\frac{1}{2}\cdot\frac{2}{\pi}}\left(\frac{\pi}{4}+\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{\sqrt{\pi}}{2}$$

and then the *sandwich theorem*

$$\lim_{n\to+\infty} \sqrt{n}\,J_n = \frac{\sqrt{\pi}}{2} \leqslant \int_0^{+\infty} e^{-x^2}dx \leqslant \frac{\sqrt{\pi}}{2} = \lim_{n\to+\infty} \sqrt{n}\,K_n$$

applied to the inequality (4b), which gives the result of the *Gaussian Integral*.

GRAZIE A TUTTI PER L'ATTENZIONE