Accidental Degeneracy of an Elliptic Differential Operator: a Clarification in Terms of Ladder Operators

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Accidental Degeneracy of an Elliptic Differential Operator: A Clarification in Terms of Ladder Operators

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Abstract: We consider the linear, second-order elliptic, Schrödinger-type differential operator $\mathcal{L} := -\frac{1}{2}\nabla^2 + \frac{r^2}{2}$. Because of its rotational invariance, that is it does not change under $SO(3)$ transformations, the eigenvalue problem $\left[-\frac{1}{2}\nabla^2 + \frac{r^2}{2}\right] f(x,y,z) = \lambda f(x,y,z)$ can be studied more conveniently in spherical polar coordinates. It is already known that the eigenfunctions of the problem depend on three parameters. The so-called *accidental degeneracy* of $\mathcal{L}$ occurs when the eigenvalues of the problem depend on one of such parameters only. We exploited ladder operators to reformulate accidental degeneracy, so as to provide a new way to describe degeneracy in elliptic PDE problems.

Keywords: degeneracy; elliptic PDE; ladder operator; commuting operator; eigenvalues

Outline

1. Introduction

Outline

Outline

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2. Eigenvalue problem
3. Ladder operators

Outline

Outline

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Elliptic operator

We consider the linear, second order elliptic differential operator

$$\mathcal{L} := -\frac{1}{2}\boldsymbol{\nabla}^2 + \frac{r^2}{2} \tag{1}$$

defined on the Hilbert space

$$\mathcal{H} = \left\{ f \in L^2(\mathbb{R}^3) \cap C^2(\mathbb{R}^3) \ : \ \lim_{r\to\infty} f(x,y,z) = 0 \right\}, \tag{2}$$

where $\boldsymbol{\nabla}^2$ denotes the *Laplacian operator* Polar Laplacian

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and r is the *norm* $r = \sqrt{x^2 + y^2 + z^2}$ of the vector $\boldsymbol{r} = (x, y, z)$.

Eigenvalue problem of the operator $\mathcal{L}$

The eigenvalue problem of the operator $\mathcal{L}$ is

$$\left[-\frac{1}{2}\nabla^2 + \frac{r^2}{2}\right] f(x,y,z) = \lambda f(x,y,z), \tag{3}$$

but since the operator $\mathcal{L}$ has rotational invariance, it can be studied more conveniently in spherical polar coordinates.

Polar eigenvalue equation

Rotational invariance

The operator $\mathcal{L}$ in (1) has the property of rotational invariance, that is, if the matrix R is a tridimensional *rotation* and we consider the transformation $\boldsymbol{r}' = (x', y', z') = R\boldsymbol{r}$, the form invariance of the operator

$$\mathcal{L}' = -\frac{1}{2}\boldsymbol{\nabla}'^2 + \frac{r'^2}{2} = -\frac{1}{2}\boldsymbol{\nabla}^2 + \frac{r^2}{2} = \mathcal{L},$$

follows, because a *rotation* R satisfies the condition

$$R^T R = RR^T = \mathbb{I}.$$

Spherical polar coordinates

Since the operator $\mathcal{L}$ in (1) has the rotational invariance, it is more convenient to study its eigenvalue problem in spherical polar coordinates given by

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$

with the conditions

$$r \in [0, \infty), \qquad \theta \in [0, \pi], \qquad \varphi \in [0, 2\pi].$$

Spherical polar coordinates

By inversion, we get the polar variables

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos(z/r) \\ \varphi = \arctan(y/x) \end{cases}$$

and if we apply the well-known Leibnitz *chain formula*

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial z}\frac{\partial}{\partial \varphi},$$

Operators in spherical polar coordinates

the *Laplacian operator* Cartesian Laplacian assumes the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\,\frac{\partial}{\partial r} - \frac{A(\theta,\varphi)}{r^2}\,, \tag{4a}$$

where we have set

$$A(\theta,\varphi) := -\frac{\partial^2}{\partial\theta^2} - \frac{\cos\theta}{\sin\theta}\,\frac{\partial}{\partial\theta} + \frac{(\mathbb{M}_3)^2}{\sin^2\theta} \tag{4b}$$

and

$$\mathbb{M}_3 = -\,i\,\frac{\partial}{\partial\varphi}\,. \tag{4c}$$

Analysis of L, A, M_3

Eigenvalue problem of the operator $\mathcal{L}$ in spherical coordinates

Since the operator $\mathcal{L}$ has rotational invariance, its associated eigenvalue problem in cartesian form Cartesian eigenvalue equation can be studied more conveniently in spherical polar coordinates, that is

$$\frac{1}{2}\left[-\frac{\partial^2}{\partial r^2}-\frac{2}{r}\frac{\partial}{\partial r}+\frac{A(\theta,\varphi)}{r^2}+r^2\right]\psi(r,\theta,\varphi)=\lambda\psi(r,\theta,\varphi),$$

on the modified Hilbert space

$$\tilde{\mathcal{H}}=\left\{f:\mathbb{R}^3\longrightarrow\mathbb{R}\ :\ \psi(r,\theta,\varphi+2\pi)=\psi(r,\theta,\varphi),\ \lim_{r\to\infty}f=0\right\}.$$

Eigenvalue problem of the operator $\mathcal{L}$

It is a standard fact that an operator A acting on a vector space of finite dimension, that is a matrix A, has at most as many eigenvalues as its order, that one can find from the equation

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

Since the eigenvalue problem

$$\frac{1}{2}\left[-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{A(\theta,\varphi)}{r^2} + r^2\right]\psi(r,\theta,\varphi) = \lambda\psi(r,\theta,\varphi) \quad (5)$$

is defined on a Hilbert space of infinite dimension, it follows that there are infinite eigenvalues λ, which form a countably infinite set of *rational numbers*.

Eigenvalues end eigenfunctions of the operator $\mathcal{L}$

The *spectrum* of the operator $\mathcal{L}$ is then given by the countably infinite set of eigenvalues Meaning of nat degen Vect $\boldsymbol{v}$ Accident degen Eigenvalues of A

$$\lambda \equiv \lambda_n = n + \frac{3}{2} \tag{6}$$

and the corresponding eigenfunctions $\psi(r, \theta, \varphi)$ are Begin L

$$\psi(r, \theta, \varphi) \equiv \psi_{n\ell m}(r, \theta, \varphi) = R_{n\ell}(r) Y_{\ell m}(\theta, \varphi), \tag{7}$$

with the following conditions on the three parameters n, ℓ, m

- n is every non-negative integer number $n = 0, 1, 2, 3, \dots$
- ℓ is every non-negative integer number less than or equal to n, having the same parity as n Actions of T1 on ψ
- m is every integer number such that $-\ell \leqslant m \leqslant \ell$.

Degeneracy of the spectrum and statemento of the problem

Since for every n there are

$$d_n = \frac{(n+1)(n+2)}{2}$$

linearly independent eigenfunctions $\psi_{n\ell m}(r,\theta,\varphi)$ associated to the eigenvalue λ_n, we say that the *spectrum* of the operator

$$\mathcal{L} = \frac{1}{2}\left\{-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{A(\theta,\varphi)}{r^2} + r^2\right\}$$

has a *degeneracy*. Begin analysis of L

The aim of our paper is to explain and to clarify a particular type this *degeneracy* through the so called *ladder operators*.

Foundamental theorem for a *Ladder Operator*

First of all, we introduce for two given operators A, B, the notation

$$[A, B] = AB - BA,$$

called *commutator* of the operators A, B.

Ladder operat example

Delta eigenvalue A by T_2 in accident degen

Theorem (shift theorem)

Let us consider an operator $\mathbb{O}$, acting on a Hilbert space, having an eigenfunction $\boldsymbol{v}$ with eigenvalue λ. If another operator $\mathbb{T}$ satisfies the condition $[\mathbb{O}, \mathbb{T}]\boldsymbol{v} = \mu\mathbb{T}\boldsymbol{v}$, where the coefficient μ is a real number, then it follows that either the function $\mathbb{T}\boldsymbol{v}$ is the null function or it is another eigenfunction of the operator $\mathbb{O}$ with eigenvalue $\lambda + \mu$.

Concept of *Ladder Operator*

Definition

An operator $\mathbb{T}$ satisfying the hypothesis of the *shift theorem* is called *ladder operator* for the operator $\mathbb{O}$ and, in particular, *lowering operator* or *raising operator*, if the coefficient μ is negative or positive, respectively.

Commuting operators

Theorem (of two commuting operators)

If $\mathbb{O}_1, \mathbb{O}_2$ are two diagonalizable operators acting on a Hilbert space such that the equality

$$[\mathbb{O}_1, \mathbb{O}_2] = 0$$

holds, then there exists a basis of the Hilbert space given by a set of common eigenvectors of both operators $\mathbb{O}_1, \mathbb{O}_2$.

Important remark

If $[\mathbb{O}_1, \mathbb{O}_2] = \mathbb{O}_1\mathbb{O}_2 - \mathbb{O}_2\mathbb{O}_1 = 0$ and $\mathbb{O}_1 \boldsymbol{v} = \lambda \boldsymbol{v}$ hold, it then follows that $\mathbb{O}_2 \boldsymbol{v}$ is either the null vector or another eigenvector of $\mathbb{O}_1$ with respect to the same eigenvalue λ as $\boldsymbol{v}$.

LadderOperatExam | Action of T2 on ψ | Eigenval of $A(\theta, \varphi)$

Example of operator having spectrum with degeneracy

Let us consider the operator described by the following matrix

$$\mathbb{O}_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The matrix $\mathbb{O}_1$ has the *simple eigenvalue* $\lambda = 0$, associated to the eigenvector $\boldsymbol{v}_0 = (0, 1, -1, 0)$, and the eigenvalue $\lambda = 2$ with *algebraic multiplicity* 3, to which the tridimensional *eigenspace*

$$E(2) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,:\, x_2 - x_3 = 0\}$$

is associated.

Example of operator having spectrum with degeneracy

Since the *eigenspace* $E(2)$ has a *degeneracy*, that is there is an "ambiguity" in the choice of its eigenvectors, we consider a second operator

$$\mathbb{O}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

commuting with $\mathbb{O}_1$, such that there exists a basis of $\mathbb{R}^4$ formed by common eigenvectors of both $\mathbb{O}_1, \mathbb{O}_2$ without any ambiguity.

Example of operator having spectrum with degeneracy

The operator $\mathbb{O}_1$ has the two simple eigenvalues $\lambda = \pm 1$ and the eigenvalue $\lambda = 0$ with algebraic multiplicity 2.

Notation as ψ

The basis formed by four common eigenvectors of $\mathbb{O}_1, \mathbb{O}_2$ is

$$\boldsymbol{v}_{0,0} = (0, 1, -1, 0),$$
$$\boldsymbol{v}_{2,-1} = (0, 0, 0, 1), \qquad \boldsymbol{v}_{2,0} = (0, 1, 1, 0), \qquad \boldsymbol{v}_{2,1} = (1, 0, 0, 0),$$

Ladd operat on this eigenvectors where the notation $\boldsymbol{v}_{\lambda_1,\lambda_2}$ has the following meaning

$$\mathbb{O}_1 \boldsymbol{v}_{\lambda_1,\lambda_2} = \lambda_1 \boldsymbol{v}_{\lambda_1,\lambda_2} \qquad \text{and} \qquad \mathbb{O}_2 \boldsymbol{v}_{\lambda_1,\lambda_2} = \lambda_2 \boldsymbol{v}_{\lambda_1,\lambda_2}.$$

In this case one says that the *degeneracy* in the spectrum of the operator $\mathbb{O}_1$ has been removed.

Clarification of the degeneracy of $\mathbb{O}_1$ with ladder operators

If we now consider the two operators

$$\mathbb{T}_+ = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{T}_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

we obtain the following relations

$$1)\ [\mathbb{O}_1, \mathbb{T}_+] = [\mathbb{O}_1, \mathbb{T}_-] = 0,$$

$$2)\ [\mathbb{O}_2, \mathbb{T}_+] = \mathbb{T}_+,$$

$$3)\ [\mathbb{O}_2, \mathbb{T}_-] = -\mathbb{T}_-\,.$$

Clarification of the degeneracy of $\mathbb{O}_1$ with ladder operators

By virtue of the *shift theorem* Shift theorem, the operators $\mathbb{T}_+, \mathbb{T}_-$ are the *ladder operators* for the operator $\mathbb{O}_2$, the *raising* and the *lowering operator*, respectively, whereas by virtue of the remark Commuting operat, the action of $\mathbb{T}_+, \mathbb{T}_-$ on the eigenvectors of $\mathbb{O}_1$ gives either the null vector or another eigenvector of $\mathbb{O}_1$ with respect to the same eigenvalue. Their action on the four basis eigenvectors Basis eigenvectors is then

$$\mathbb{T}_+ \boldsymbol{v}_{0,0} = \mathbb{T}_- \boldsymbol{v}_{0,0} = \mathbf{0},$$

$$\mathbb{T}_+ \boldsymbol{v}_{2,-1} = \boldsymbol{v}_{2,0}, \qquad \mathbb{T}_+ \boldsymbol{v}_{2,0} = 2\boldsymbol{v}_{2,1}, \qquad \mathbb{T}_+ \boldsymbol{v}_{2,1} = \mathbf{0},$$

$$\mathbb{T}_- \boldsymbol{v}_{2,1} = \boldsymbol{v}_{2,0}, \qquad \mathbb{T}_- \boldsymbol{v}_{2,0} = 2\boldsymbol{v}_{2,-1}, \qquad \mathbb{T}_- \boldsymbol{v}_{2,-1} = \mathbf{0}.$$

The operator $\mathcal{L}$ and its associated operators

The operator $\mathcal{L}$ that we are studying Definition of L, given by

$$\mathcal{L} = \frac{1}{2}\left\{-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{A(\theta,\varphi)}{r^2} + r^2\right\}$$

and having the eigenfunctions Eigenfunctions of L

$$\psi_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)\,Y_{\ell m}(\theta,\varphi),$$

contains the two associated operators Definition of A,M3

$$A(\theta,\varphi) := -\frac{\partial^2}{\partial\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{(\mathbb{M}_3)^2}{\sin^2\theta} \tag{8}$$

$$\mathbb{M}_3 = -i\,(\partial/\partial\varphi), \tag{9}$$

such that $[\mathcal{L}, A(\theta,\varphi)] = [\mathcal{L}, \mathbb{M}_3] = [A(\theta,\varphi), \mathbb{M}_3] = 0.$

Ladd operat T_2 for accident degen

The operator $\mathcal{L}$ and its associated operators

The eigenvalue equations of the operators $A(\theta,\varphi)$ and $\mathbb{M}_3$ are

$$A(\theta,\varphi)\psi_{n\ell m}(r,\theta,\varphi)=\ell(\ell+1)\psi_{n\ell m}(r,\theta,\varphi),$$

$$\mathbb{M}_3\psi_{n\ell m}(r,\theta,\varphi)=m\psi_{n\ell m}(r,\theta,\varphi).$$

Commutat theorem Shift for the accident degen

In the literature, the following *ladder operators* for $\mathbb{M}_3$

$$\mathbb{T}_1^{(+)}:=e^{i\varphi}\left(\frac{\partial}{\partial\theta}+\frac{i\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right),$$

$$\mathbb{T}_1^{(-)}:=e^{-i\varphi}\left(\frac{i\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}-\frac{\partial}{\partial\theta}\right)$$

Eigenfunctions ψ Meaning of the nat degen

Ladder operators of the so called *natural degeneracy*

explain and clarify the so called *natural degeneracy* of the operator $\mathcal{L}$, because it yields

$$\left[\mathcal{L}, \mathbb{T}_1^{(\pm)}\right] = 0, \qquad \left[A(\theta,\varphi), \mathbb{T}_1^{(\pm)}\right] = 0, \qquad \left[\mathbb{M}_3, \mathbb{T}_1^{(\pm)}\right] = \pm\mathbb{T}_1^{(\pm)},$$

from which the actions of the *ladder operators* on the eigenfunctions $\psi_{n,\ell,m}(r,\theta,\varphi)$ of $\mathcal{L}$

$$\mathbb{T}_1^{(\pm)}\psi_{n,\ell,m}(r,\theta,\varphi) = C\,\psi_{n,\ell,m\pm1}(r,\theta,\varphi).$$

follow.

Ladder operators of the so called *natural degeneracy*

The iteration of these actions is

$$\begin{aligned}
&\mathbb{T}_1^{(+)}\psi_{n,\ell,-\ell}(r,\theta,\varphi) = C\psi_{n,\ell,-\ell+1}(r,\theta,\varphi),\\
&\mathbb{T}_1^{(+)}\psi_{n,\ell,-\ell+1}(r,\theta,\varphi) = C\psi_{n,\ell,-\ell+2}(r,\theta,\varphi),\\
&\vdots\\
&\mathbb{T}_1^{(+)}\psi_{n,\ell,\ell-1}(r,\theta,\varphi) = C\psi_{n,\ell,\ell}(r,\theta,\varphi),
\end{aligned}$$

in the direction *bottom-up*, or

$$\begin{aligned}
&\mathbb{T}_1^{(-)}\psi_{n,\ell,\ell}(r,\theta,\varphi) = C\psi_{n,\ell,\ell-1}(r,\theta,\varphi),\\
&\mathbb{T}_1^{(-)}\psi_{n,\ell,\ell-1}(r,\theta,\varphi) = C\psi_{n,\ell,\ell-2}(r,\theta,\varphi),\\
&\vdots\\
&\mathbb{T}_1^{(-)}\psi_{n,\ell,-\ell+1}(r,\theta,\varphi) = C\psi_{n,\ell,-\ell}(r,\theta,\varphi),
\end{aligned}$$

in the direction *up-bottom*. Conditions on n,l,m

Meaning of the *natural degeneracy*

In other words, the *natural degeneracy* is the independence of the eigenvalue λ_n from the parameter m Eigenvalue di L.

This kind of *degeneracy* is due to and explained by the existence of the couple of *ladder operators* $\mathbb{T}_1^{(\pm)}$ Ladder operators T1.

Meaning of the *accidental degeneracy*

The so called *accidental degeneracy* is the independence of the eigenvalue λ_n also from the parameter ℓ Eigenvalue di L.

This kind of *degeneracy* is due to and explained by the existence of the couple of *ladder operators* $\mathbb{T}_2^{(\pm)}$ for the operator $A(\theta, \varphi)$ given by

$$\mathbb{T}_2^{(\pm)} := -\frac{\partial}{\partial x}\frac{\partial}{\partial y} + xy \pm \frac{i}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + y^2 - x^2\right),$$

whose commutation rules with the operators $\mathcal{L}, A(\theta, \varphi), \mathbb{M}_3$ are the following

L, A, M_3 for commutat with T_2

Commutation rules for the *accidental degeneracy*

$$\left[\mathcal{L}, \mathbb{T}_2^{(+)}\right] = 0, \qquad \left[A(\theta,\varphi), \mathbb{T}_2^{(+)}\right] \approx (4\ell+6)\mathbb{T}_2^{(+)},$$

$$\left[\mathbb{M}_3, \mathbb{T}_2^{(+)}\right] = 2\mathbb{T}_2^{(+)},$$

from which we get the action of the *ladder operator* $\mathbb{T}_2^{(+)}$ on the eigenfunctions $\psi_{n,\ell,m}(r,\theta,\varphi)$ of $\mathcal{L}$

Remark on commuting operat

$$\mathbb{T}_2^{(+)}\psi_{n,\ell,\ell} = C\,\psi_{n,\ell+2,\ell+2},$$

where it yields Shift theorem Eigenvalues of A shifted

$$4\ell+6 = [(\ell+2)(\ell+3)] - [\ell(\ell+1)].$$

Actions of the ladder operators for the *accidental degeneracy*

The iterated actions of the *ladder operators* $\mathbb{T}_2^{(\pm)}$ are:

for every even number n

$$\begin{cases} \mathbb{T}_2^{(+)}\psi_{n,0,0} = C\,\psi_{n,2,2}\,, \\ \mathbb{T}_2^{(+)}\psi_{n,2,2} = C\,\psi_{n,4,4}\,, \\ \vdots \\ \mathbb{T}_2^{(+)}\psi_{n,n-2,n-2} = C\,\psi_{n,n,n}\,, \end{cases}$$

and analogously

for every odd number n

$$\begin{cases} \mathbb{T}_2^{(+)}\psi_{n,1,1} = C\,\psi_{n,3,3}\,, \\ \mathbb{T}_2^{(+)}\psi_{n,3,3} = C\,\psi_{n,5,5}\,, \\ \vdots \\ \mathbb{T}_2^{(+)}\psi_{n,n-2,n-2} = C\,\psi_{n,n,n}\,. \end{cases}$$

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