The Shapley Value and the Banzhaf Value as Instruments for Portfolio Selection

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Basic motivation I

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Can instruments borrowed from Cooperative Game Theory such as the Shapley value and the Banzhaf value be employed as a criterion to select portfolios?

Basic motivation II

In recent literature, the Shapley value has been used to assess the marginal contribution of each asset to the overall volatility of the portfolio (Shalit, 2020 and 2021) in the framework of a risk game. In such a game, the values of the portfolios, which are identified with the subsets of the set of available assets, are their volatilities.

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Namely, we are going to rely on volatility in the Markowitz problems, and on expected returns in the cooperative game.

Our strategy I

Consider the set $\mathcal{N}$ of N risky assets (or securities) in the standard Markowitz setting.
The i-th asset is identified by $A_i = (\sigma_i,\ M_i)$, where $\sigma_i \geq 0$ is the volatility and $M_i \geq 0$ is the expected return.

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Here is our strategy:

1. firstly, we consider all the available subsets of the set of assets, which are $2^N - N - 1$. Then we apply Markowitz' minimization of variance to every subset, thereby obtaining the minimum variance portfolios (MVPs) for every possible collection of assets. The sum of the weights of the assets amounts to 1, and this is the unique constraint to which the problem is subject.

Our strategy II

1. Subsequently, we calculate the returns of all the MVPs that have been determined initially so as to associate a positive value to all the collections of assets.
2. The assignment of the return value to each possible subset of the set of assets naturally induces a cooperative game, which turns out to be a payoff game.
3. Calculating the Shapley and the Banzhaf values of the above cooperative game yields a ranking among assets, which can be also employed as a preference structure.

Determination of MVPs and construction of the payoff game I

Here we summarize the notation and outline the construction of our game.

- $p = (p_1, \ldots, p_N)$ is the vector of portfolio weights, subject to the standard linear constraint $p_1 + \cdots + p_N = 1$. If we assume that no short sales are allowed, $p_i \in [0, 1]$ for every $i = 1, \ldots, N$.

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- C is the $N \times N$ variance-covariance matrix.

Determination of MVPs and construction of the payoff game II

- $\mathcal{P}^*$ is the MVP which is usually calculated by minimizing the quadratic form $\sigma^2(p_1, \ldots, p_N) = < p, (Cp^T)^T >$ subject to the above constraint. If $(p_1^*, \ldots, p_N^*)$ are the weights of the MVP, whose volatility is σ^*, we have

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- $v : 2^N \longrightarrow \mathbb{R}$ is the function which maps each portfolio of assets to the return level of that portfolio of assets. Hence, $(v,\ \mathcal{N})$ is a cooperative game provided that $v(\emptyset) = 0$, i.e. there is no return if the portfolio is empty, because no investment is carried out.

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Hence, we define the payoff game as follows:

$$v(S) = \sum_{i \in S} p_i^* M_i,$$

except for the singletons, i.e. $v(\{A_i\}) = M_i$, and for the empty set: $v(\emptyset) = 0$.

The preference scheme induced by the Shapley value and by the Banzhaf value I

Definition

Given 2 assets $A_i = (\sigma_i,\ M_i),\ A_j = (\sigma_j,\ M_j) \in \mathcal{N}$, *we say that* A_i **is weakly Shapley-preferred to** A_j *and we write* $A_i \succeq_\Phi A_j$ *if* $\Phi_i(v^*) \geq \Phi_j(v^*)$.

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The preference scheme induced by the Shapley value and by the Banzhaf value II

Some results on the preference scheme

Proposition

In a 3-player game $(v^*, \mathcal{N})$, $A_i \succeq_\Phi A_j$ *and* $A_i \succeq_\beta A_j$ *if and only if*

$$v^*(\{A_i\}) - v^*(\{A_j\}) + v^*(\{A_i, A_k\}) - v^*(\{A_j, A_k\}) \geq 0.$$

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Proposition

Given A_i, A_j and A_k in a 3-asset game $(v^*, \mathcal{N})$, if

$$M_i > \max\left\{M_j, \frac{\sigma_i^2}{\sigma_j^2}(M_j - M_k) + M_k\right\},$$

then $A_i \succ_\Phi A_j$ and $A_i \succ_\beta A_j$.

A 3-assets numerical example I

Consider the 3 risky assets which are identified by the following points in the risk-return plane:

$$A_1 = (0.04,\ 0.03),\ A_2 = (0.05,\ 0.04),\ A_3 = (0.06,\ 0.07),$$

and the related covariances are

$$\sigma_{12} = -0.02, \qquad \sigma_{13} = 0, \qquad \sigma_{23} = 0.$$

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The portfolios which are composed of only one asset are trivial, in that the volatilities of the related MVPs are the volatilities of the assets:

$$v^*(\{A_1\}) = 0.03; \qquad v^*(\{A_2\}) = 0.04; \qquad v^*(\{A_3\}) = 0.07.$$

A 3-assets numerical example II

Firstly, we have to calculate the MVPs of each 2-assets portfolio. Consider the portfolio $\{A_1,\ A_2\}$, where we have the following variance expression:

$$\sigma^2(p_1, p_2) = 0.04^2 p_1^2 + 0.05^2 p_2^2 - 0.04 \cdot 0.04 \cdot 0.05 p_1 p_2,$$

that is supposed to be minimized subject to the constraint $p_1 + p_2 = 1$. The MVP is $(p_1^*, p_2^*) = (0.631,\ 0.369)$, and the related expected return is $v^*(\{A_1, A_2\}) = 0.0337$.

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Considering assets A_1 and A_3 yields the variance function:

$$\sigma^2(p_1, p_3) = 0.04^2 p_1^2 + 0.06^2 p_3^2 = 0.04^2 p_1^2 + 0.06^2 (1 - p_1)^2,$$

which attains its minimum level at $(p_1^*, p_3^*) = (0.692,\ 0.308)$, consequently

$$v^*(\{A_1, A_3\}) = 0.0423.$$

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The last MVP to be calculated is the one with all the assets, for which the Lagrange's multipliers method is necessary. The variance function which must be minimized reads as

$$\sigma^2(p_1, p_2, p_3) = 0.04^2 p_1^2 + 0.05^2 p_2^2 +$$

$$+0.06^2 p_3^2 - 0.04 \cdot 0.04 \cdot 0.05 p_1 p_2,$$

subject to the linear constraint $p_1 + p_2 + p_3 = 1$.

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subject to the linear constraint $p_1 + p_2 + p_3 = 1$.
Solving the constrained problem yields the MVP

$$(p_1, p_2, p_3) = (0.55,\ 0.347,\ 0.103),$$

A 3-assets numerical example IV

then we can deduce the last value for $v^*(\cdot)$:

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Since we have all the relevant values now, we can calculate the Shapley and Banzhaf indices of the game $(v^*, \{A_1, A_2, A_3\})$.

A 3-assets numerical example V

$$\Phi_1(v^*) = \frac{0!2!}{3!}[v^*(\{A_1, A_2, A_3\}) - v^*(\{A_2, A_3\})] +$$

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$$+\frac{1!1!}{3!}[v^*(\{A_1, A_3\}) - v^*(\{A_3\})] + \frac{2!0!}{3!}[v^*(\{A_1\}) - v^*(\emptyset)] =$$

$$= \frac{1}{3}(0.0376 - 0.0523) +$$

$$+\frac{1}{6}(0.0337 - 0.04) + \frac{1}{6}(0.0423 - 0.07) + \frac{1}{3} \cdot 0.03 = -0.0005.$$

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By analogous arguments, we obtain

$$\Phi_2(v^*) = 0.0094, \qquad \Phi_3(v^*) = 0.0287.$$

A 3-assets numerical example VI

This means that A_3 contributes more than the other assets to the overall payoff of the complete portfolio, which is quantified as $v^*(\{A_1, A_2, A_3\}) = 0.0376$. Note that the contribution of A_1 is negative, meaning that if A_1 were not included among the players, the total gain would be higher (namely 0.0523).

A 3-assets numerical example VI

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$$\beta_1(v^*) = \frac{1}{4}\left[0.0376 - 0.0523 + 0.0337 - 0.04 + 0.0423 - 0.07 + 0.03\right] = -0.0046.$$

$$\beta_2(v^*) = \frac{1}{4}\left[0.0376 - 0.0423 + 0.0337 - 0.03 + 0.0523 - 0.07 + 0.04\right] = 0.0053.$$

$$\beta_3(v^*) = \frac{1}{4}\left[0.0376 - 0.0337 + 0.0423 - 0.03 + 0.0523 - 0.04 + 0.07\right] = 0.0246.$$

A 3-assets numerical example VII

In both cases the chain of inequalities remains the same:

$$\Phi_1(v^*) < \Phi_2(v^*) < \Phi_3(v^*), \quad \beta_1(v^*) < \beta_2(v^*) < \beta_3(v^*).$$

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Finally, note that both chains exactly reproduce the ordering induced by the assets' returns, i.e.

$$M_1 < M_2 < M_3.$$

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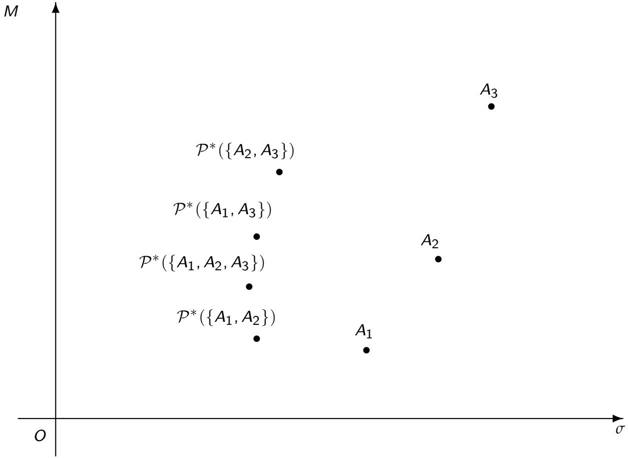
All the MVPs we obtained are summarized here:

$$\mathcal{P}^*(\{A_1, A_2, A_3\}) = (0.0284,\ 0.0376), \qquad \mathcal{P}^*(\{A_1, A_2\}) = (0.031,\ 0.0337),$$

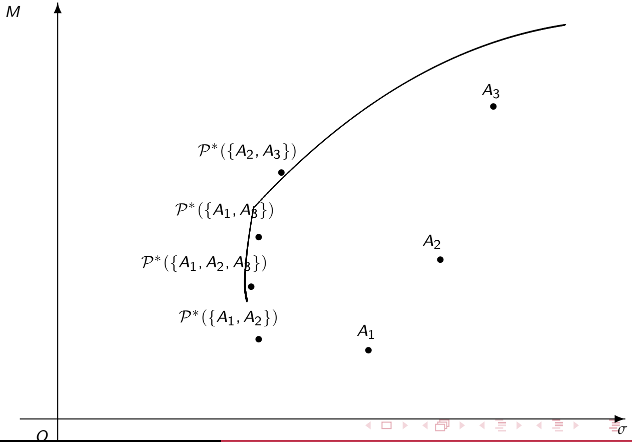
$$\mathcal{P}^*(\{A_1, A_3\}) = (0.0332,\ 0.0423), \qquad \mathcal{P}^*(\{A_2, A_3\}) = (0.0384,\ 0.0523).$$

A 3-assets numerical example VIII



A 3-assets numerical example IX

A kind of efficient frontier for the found portfolios?



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As is well known from standard Markowitz theory, not all portfolios are comparable with respect to both dimensions, such as $\mathcal{P}^*(\{A_1, A_3\})$ and $\mathcal{P}^*(\{A_2, A_3\})$.
But since A_2 is Shapley-preferred and Banzhaf-preferred to A_1, we have a criterion to select $\mathcal{P}^*(\{A_2, A_3\})$.

Ideas for possible extensions

- Identification of further properties of the preference schemes induced by the Shapley value and the Banzhaf value.
- Extension of this setup to different kinds of portfolio optimization problems.
- Analysis of the compliance of this preference with the standard preference schemes.
- Determination of a procedure to test this method with real financial data.

Some necessary references

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