Regularity of the Volatility Process Using the Lamperti Transform

S. Bianchi[a] D. Angelini[a] M. Frezza[a] A. Pianese[b]

[a]MEMOTEF, University "La Sapienza" Rome, Italy

[b]QUANTLAB, University of Cassino and Southern Lazio, Italy

27 June 2023

XIII Giornate della Ricerca MEMOTEF, 27-28 June 2023

Contact: sergio.bianchi@uniroma1.it

Table of Contents

Table of Contents

Preliminaries: Self-similarity

Definition (Self-similar stochastic process)

The $\mathbb{R}^k-$valued stochastic process $\{X_t, t \geq 0\}$, nontrivial and continuous at $t = 0$, is self-similar with parameter $H_0 \geq 0$ (H_0-ss) if for any $a > 0$,

$$\{X_{at}\} \stackrel{d}{=} \{a^{H_0} X_t\} \tag{1}$$

where $\stackrel{d}{=}$ denotes the equality of the k-dimensional distributions of X_t.

Remark

If X_t has stationary increments, then they are also self similar with the same parameter H_0 (H_0-sssi), that is:

$$\{X_{t+a} - X_t\} \stackrel{d}{=} \{a^{H_0}(X_{t+1} - X_t)\}. \tag{2}$$

Preliminaries: the Lamperti transform

The key role of the work is played by the following

Theorem (Lamperti (1962))

If $\{X_t, t \geq 0\}$ *is H-ss and*

$$V_t = e^{-tH} X_{e^t}, t \in \mathbb{R}, \tag{3}$$

then $\{V_t, t \in \mathbb{R}\}$ *is strictly stationary.*

Conversely, if $\{V_t, t \in \mathbb{R}\}$ *is a strictly stationary process and if for some* $H > 0$

$$X_t = t^H V_{\log t}, \text{ for } t > 0; \; X_0 = 0, \tag{4}$$

then $\{X_t, t \geq 0\}$ *is H-ss.*

Preliminaries: (Fractional) Ornstein-Uhlenbeck process

Consider the linear SDE (Langevin-like equation) driven by a fractional Brownian motion (fBm) of parameter H_0

$$dV_t^{H_0} = -\alpha V_t dt + dB_t^{H_0}, \quad t \geq 0, \tag{5}$$

with $\alpha > 0$.

The solution of equation (5) is

$$V_t^{H_0} = e^{-\alpha t}\left(V_0 + \int_0^t e^{\alpha s} dB_s^{H_0}\right) \tag{6}$$

where the initial value $V_0^{H_0} := \int_{-\infty}^0 e^{\alpha s} d\widehat{B}_s^{H_0}$ with $\widehat{B}_t^{H_0}$ a two-sided fBm.

In general, for every $V_0^{H_0} \in L^0(\Omega)$, $V_t^{H_0}$ is called a fractional Ornstein-Uhlenbeck process (fOU) with initial condition $V_0^{H_0}$.

Properties I

- fOU is a stationary process
- When $H_0 = 1/2$ the fOU process reduces to the classical Ornstein–Uhlenbeck process
- The Ornstein–Uhlenbeck process can be obtained from Brownian motion by Lamperti transform

$$V_t^{1/2} = e^{-t}\left(V_0^{1/2} + 2^{-1/2} B_{e^{2t}-1}\right)$$

in the sense of having the same finite dimensional distributions.

- Property above does not hold for the fOU if $H_0 \neq 1/2$, because $V_t^{H_0}$ in (6) has not the same finite dimensional distribution as its Lamperti transform

$$Z_t^{H_0} = e^{-t}\left(V_0^{H_0} + \sqrt{H_0} B^H_{e^{t/H_0}-1}\right), \quad t \geq 0$$

Properties II

- The pathwise Riemann-Stieltjes integral in (6) exists and, for $H \in (0, 1/2) \cup (1/2, 1]$, $N = 1, 2, \ldots$, $t \in \mathbb{R}$ fixed and $s \to \infty$ it is [Cheridito, Kawaguchi, and Maejima (2003)]

$$\mathrm{Cov}(\mathrm{V}_t^{H_0}, V_{t+s}^{H_0}) = \frac{1}{2} \sum_{n=1}^{N} \alpha^{-2n} \left(\prod_{k=0}^{2n-1} (2H_0 - k) \right) s^{2(H_0 - n)} + \mathcal{O}\left(s^{2(H_0 - N - 1)} \right)$$

- $\mathrm{Cov}(V_t^{H_0}, V_{t+s}^{H_0})$ decays similarly to $\mathrm{Cov}(B_{h+t}^{H_0} - B_h^{H_0}, B_{h+s+t}^{H_0} - B_{h+s}^{H_0})$ and exhibits short-range dependence if $H < 1/2$ and long-range dependence if $H > 1/2$.

Properties III

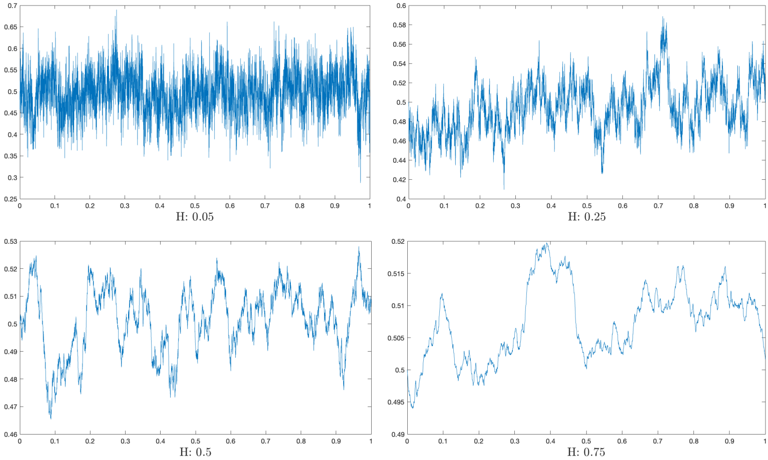

Figure: Simulated paths of fractional Ornstein-Uhlenbeck process with different Hurst exponents ($\alpha = 0.01; m = 0.5, \rho = 0.3$)

Table of Contents

(Rough) Fractional Stochastic Volatility Model

Comte and Renault's Fractional Stochastic Volatility model (1998)

The stock price S_t is described by the following system of SDE's

$$\begin{cases} dS_t = \mu(t, S_t)dt + S_t \sigma_t dW_t \\ d\log(\sigma_t) = \alpha[m - \log(\sigma_t)]dt + \rho dB_t^H \end{cases} \tag{7}$$

- The log-volatility ($\log \sigma_t$) is modeled by a fOU process.
- Gatheral, Jaisson, and Rosenbaum (2014); Gatheral, Jaisson, and Rosenbaum (2018) provide empirical evidence of $H \sim 0.1$, which turns (7) into Rough Fractional Stochastic Volatility (RFSV).
- Generalizations: Rough Bergomi model [Bayer, Friz, and Gatheral (2016)]; Rough Heston model [El Euch, Gatheral, and Rosenbaum (2019)].

Estimation of the regularity parameter, I

Pro roughness

- Generalized Hurst Exponent
  Livieri et al. (2018): $H \simeq 0.3$ for ATM options S&P500
- Quasi-Likelihood Estimation
  Fukasawa, Takabatake, and Westphal (2019): $H \in (0.02, 0.06)$ for five 5-min stock indices.
- Generalized Method of Moments
  Bolko et al. (2022): $H \lesssim 0.05$ for a large panel of equity indices.
- OLS on second order variogram
  Bennedsen, Lunde, and Pakkanen (2021): $H \in (0.1, 0.2)$ for S&P500 E-mini futures contracts.
- Multifractal Detrended Fluctuation Analysis
  Takaishi (2020): non costant generalized Hurst exponent in Bitcoin volatility.

Estimation of the regularity parameter, II

Questionable roughness

- Cont and Das (2022) use a non-parametric method based on normalized p-th variation along a sequence of partitions and show that the realized volatility exhibits rough behaviour, irrespective of the roughness of the spot volatility process.
- Rogers (2023) finds that the roughness of volatility can be explained by simpler models such as a bivariate Ornstein-Uhlenbeck model.
- Angelini and Bianchi (2023) show that highly nonlinear biases appear when the Hurst exponent is estimated by smoothing moment-based methodologies.

Contribution

- We apply the Lamperti transform to convert the (stationary) log-volatility into a self-similar process.

  In this regard, we prove that:

  - The self-similarity parameter of the transformed process is invariant w.r.t. the parameter of the Lamperti transform
  - The Lamperti transform of the fOU process driven by a fBm of parameter H_0 is itself H_0-ss

- We use a distribution-based method [Bianchi (2004)] to estimate the self-similarity parameter of the transformed process.

  - Open problem:
    Generalize the Kolmogorov-Smirnov distributions to dependent data

- We provide evidence of roughness in the log-volatility

Main results I

We state the following

Proposition (Uniqueness)

Let $\{Y_t, t \in \mathbb{R}\}$ be strictly stationary. Then its Lamperti transform $X_t = t^H Y_{\log t}$ is H_0-ss for any H.

Implication. The Lamperti transform can be calculated with any H.

Proof. Let $H \in (0, 1]$, $X_0 = 0$ and $t > 0$. From the Lamperti transform one has

$$X_t = t^H Y_{\log t} \quad \text{and} \quad X_{at} = (at)^H Y_{\log at}.$$

By definition, X_t is self-similar of parameter H_0 if $\{a^{H_0} X_t\} \stackrel{d}{=} \{X_{at}\}$, that is

$$\{a^{H_0} t^H Y_{\log t}\} \stackrel{d}{=} \{a^H t^H Y_{\log at}\}$$

Main results II

The factor t^H can be cancelled and this entails that the choice of H in equation (4) does not affect the self-similarity parameter H_0. In fact,

$$\{a^{H_0} Y_{\log t}\} \stackrel{d}{=} \{a^H Y_{\log a + \log t}\}.$$

By the stationarity of Y_t, $\{Y_{\log a + \log t}\} \stackrel{d}{=} \{Y_{\log t}\}$, therefore

$$\{a^{H_0} Y_{\log t}\} \stackrel{d}{=} \{a^H Y_{\log t}\}$$

if and only if $H = H_0$, whatever t^H is considered in the Lamperti transform. $\square$

Main results III

Proposition (Identity)

The Lamperti transform (4) of the fOU process driven by a fBm of parameter H_0 is H_0-ss.

Implication. The roughness of the log-volatility (provided it follows a fOU process) can be estimated as the self-similarity parameter of the Lamperti transformed process.

Proof. Without loss of generality, we set $V_0 = 0$, so that

$$V_t = e^{-\alpha t} \int_0^t e^{\alpha s} dB_s^{H_0}.$$

Main results IV

For any H, we have to prove that $X_{at} \stackrel{d}{=} a^{H_0} X_t$ with $X_t = t^H V_{\log t}$. It is

$$
\begin{aligned}
X_{at} &= (at)^H V_{\log(at)} \\
&= (at)^H e^{-\alpha \log(at)} \int_0^{\log(at)} e^{\alpha s} dB_s^{H_0} \\
&= (at)^{H-\alpha} \int_0^{\log(at)} e^{\alpha s} dB_s^{H_0} \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
a^{H_0} X_t &= a^{H_0} t^H V_{\log(t)} \\
&= a^{H_0} t^H e^{-\alpha \log t} \int_0^{\log t} e^{\alpha s} dB_s^{H_0} \\
&= a^{H_0} t^{H-\alpha} \int_0^{\log t} e^{\alpha s} dB_s^{H_0}. \qquad (9)
\end{aligned}
$$

Main results V

Equating in distribution (8) and (9), we have

$$a^{H-\alpha} \int_0^{\log(at)} e^{\alpha s} dB_s^{H_0} \overset{d}{=} a^{H_0} \int_0^{\log t} e^{\alpha s} dB_s^{H_0}. \tag{10}$$

Let us prove that the above equality holds if and only if $H = H_0$. If this equality is true, then simplifying (10) we have

$$\int_0^{\log(at)} e^{\alpha s} dB_s^{H_0} \overset{d}{=} a^{\alpha} \int_0^{\log t} e^{\alpha s} dB_s^{H_0}. \tag{11}$$

Setting $v = s + \log a$ yields to

$$\int_0^{\log(at)} e^{\alpha s} dB_s^{H_0} \overset{d}{=} a^{\alpha} \int_{\log a}^{\log(at)} e^{\alpha(v-\log a)} dB_{v-\log a}^{H_0}$$

$$\int_0^{\log(at)} e^{\alpha s} dB_s^{H_0} \overset{d}{=} a^{\alpha} \int_{\log a}^{\log(at)} e^{\alpha v} a^{-\alpha} dB_{v-\log a}^{H_0}.$$

Main results VI

Since:

- stationarity of dB^{H_0} entails that $dB^{H_0}_{v-\log a} \stackrel{d}{=} dB^{H_0}_v$
- V is a stationary process as well [Cheridito, Kawaguchi, and Maejima (2003)]

we finally have

$$\int_0^{\log(at)} e^{\alpha s} dB^{H_0}_s \stackrel{d}{=} \int_0^{\log t} e^{\alpha v} dB^{H_0}_v. \tag{12}$$

Thus, process X_t is self-similar with parameter H_0. $\square$

Self-similarity through the diameter of PDF space I

Given the compact set of timescales $\mathcal{A} = [\underline{a}, \overline{a}] \subset \mathbb{R}^+$, for any $a \in \mathcal{A}$, denote by $F_{X_a}(x)$ the CDF of X_{at}.

Equation (1) can be written as

$$F_{X_a}(x) := \mathbf{P}\left(X_{at} < x\right) = \mathbf{P}\left(a^{H_0} X_t < x\right) = F_{X_1}(a^{-H_0} x), \qquad (13)$$

or, introducing the variable H, as

$$F_{a^{-H} X_a}(x) := \mathbf{P}\left(a^{-H} X_{at} < x\right) = \mathbf{P}\left(a^{H_0 - H} X_t < x\right) = F_{X_1}(a^{H - H_0} x).$$

Let:

- $\Psi_H := \{F_{a^{-H} X_a}(x), a \in \mathcal{A}, x \in \mathbb{R}\}$ be the set of CDF's of $\{a^{-H} X_{at}\}$
- consider the distance function ρ induced by the sup-norm $|| \cdot ||_\infty$ with respect to $\mathcal{A}$.

Self-similarity through the diameter of PDF space II

The diameter of the metric space (Ψ_H, ρ) is then defined as

$$\begin{aligned}
\delta(\Psi_H) &:= \sup_{x\in\mathbb{R}} \sup_{a,b\in\mathcal{A}} |F_{a^{-H}X_a}(x) - F_{b^{-H}X_b}(x)| \\
&= \sup_{x\in\mathbb{R}} \sup_{a,b\in\mathcal{A}} |F_{X_1}(a^{H-H_0}x) - F_{X_1}(b^{H-H_0}x)| \\
&= \sup_{x\in\mathbb{R}} |F_{X_1}(\underline{a}^{H-H_0}x) - F_{X_1}(\overline{a}^{H-H_0}x)|. \qquad (14)
\end{aligned}$$

where $\underline{a} = \min \mathcal{A}$ and $\overline{a} = \max \mathcal{A}$.

Self-similarity through the diameter of PDF space III

The following Propositions hold for $\delta(\Psi_H)$:

Proposition ("Zero" δ)

The process $\{X_t, t \geq 0\}$ is H_0-ss if and only if $\delta(\Psi_{H_0}) = 0$.

Proposition (Monotonicity in H)

Let $\{X_t, t \geq 0\}$ be H_0-ss. Then $\delta(\Psi_H)$ is non-increasing for $H \leq H_0$ and non-decreasing for $H \geq H_0$.

Proposition (Monotonicity in timescales)

Let $\{X_t, t \geq 0\}$ be H_0-ss and $\{\mathcal{A}_i\}_{i=1,\ldots,n}$ be a sequence of timescale sets such that $\underline{a}_i \leq \underline{a}_j$ and $\overline{a}_i \geq \overline{a}_j$ for $i > j$, then, with respect to $\{\mathcal{A}_i\}$, $\delta_X(\Psi_H)$ is: (i) non-decreasing, if $H \neq H_0$; (ii) zero, if $H = H_0$.

Self-similarity through the diameter of PDF space IV

Proposition (Diameter of fBm)

Let $\{X_t, t \geq 0\}$ be fractional Brownian motion of parameter H_0, with $\sigma^2 = \mathbb{E}(B_1^H)$. Then

$$\delta_{B_t^H}(\Psi_H) = \sup_{x\in\mathbb{R}} \frac{1}{\sigma\sqrt{2\pi}} \int_{x\overline{a}^{H-H_0}}^{x\underline{a}^{H-H_0}} e^{-u^2/2\sigma^2}\, du,$$

that is

$$\delta_{B_t^H}(\Psi_H) = \Phi(\hat{x}\overline{a}^{H_0-H}) - \Phi(\hat{x}\underline{a}^{H_0-H})$$

where Φ denotes the cumulative distribution function of a standard normal random variable and

$$\hat{x} = \begin{cases} \sqrt{\frac{2(H_0-H)}{\overline{a}^{2(H_0-H)} - \underline{a}^{2(H_0-H)}} \log \frac{\overline{a}}{\underline{a}}}, & \text{if } H \neq H_0 \\ 0, & \text{if } H = H_0. \end{cases}$$

Examples I

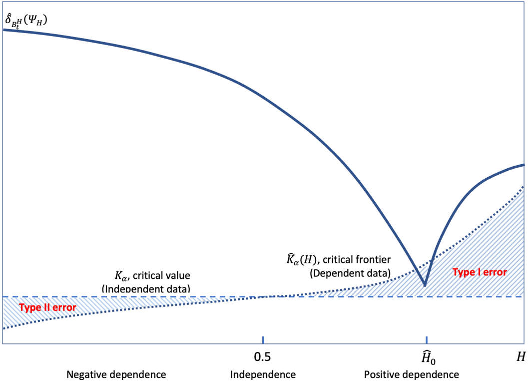

Figure: Sketched estimation of the self-similarity parameter H_0 as $\mathrm{argmin}_{H \in [0,1)} \hat{\delta}(\Psi_H)$. The critical value of the KS test refers to the independence (dashed line), whereas dependence in the samples reflects in a new frontier of critical values (dotted line). Here, assuming to refer to a B_t^H, the two frontiers intersect at the point $H = 0.5$ (independence).

Examples II

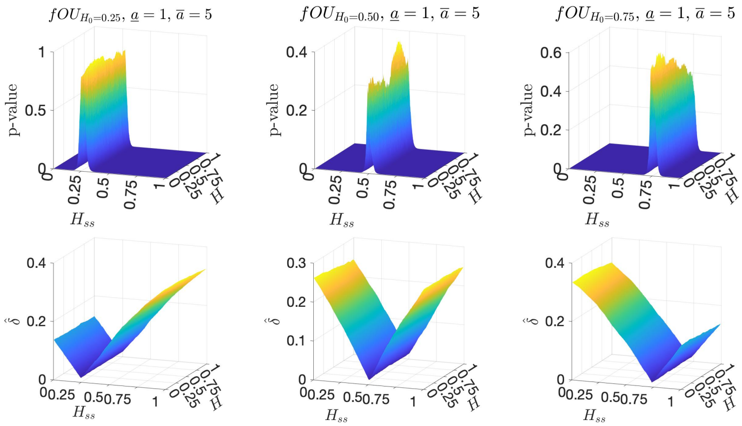

Figure: (Bottom panels) KS tests for three simulated fOU of length $n = 5,000$, with prescribed $H_0 = \{0.25, 0.5, 0.75\}$. The diameter $\hat{\delta}(\Psi_{H_{ss}})$ is reported as a function of the Lamperti transform parameter $H \in (0,1)$, with step 0.01, and of the self-similarity parameter $H_{ss} \in (0,1)$, with step 0.0001. (Top panels) p-values of the KS tests as a function of H and H_{ss}. The minimum $\hat{\delta}$ (maximum p-value) is achieved in correspondence of the self-similarity parameters $\hat{H}_0$ prescribed for the fBm driving the simulated fOU. The set of timescales used is $\mathcal{A} = [\![1,5]\!]$.

Table of Contents

Data analysis, stationarity of log-volatility

Ticker	Index	Obs.	ADF	p-value
SPX	Standard & Poor 500 (USA)	4641	−14.390	< 0.001
FTSE	Footsie 100 (GBR)	4660	−15.886	< 0.001
N225	Nikkei 225 (JPN)	4501	−16.238	< 0.001
GDAXI	Dax 30 (DEU)	4692	−14.267	< 0.001
RUT	Russell 2000 (USA)	4636	−18.909	< 0.001
AORD	All Ordinaries (AUS)	4665	−18.861	< 0.001
DJI	Dow Jones I. (USA)	4635	−15.354	< 0.001
IXIC	Nasdaq C. (USA)	4637	−13.526	< 0.001
FCHI	Cac 40 (FRA)	4713	−14.405	< 0.001
HSI	Hang Seng I. (HKG)	4532	−15.611	< 0.001
KS11	Kospi C. I. (KOR)	4550	−11.902	< 0.001
AEX	Amsterdam Exchange I. (NLD)	4714	−13.850	< 0.001
SSMI	Swiss Market I. (CHE)	4636	−13.219	< 0.001
IBEX	Ibex 35 (ESP)	4681	−14.458	< 0.001
NSEI	Nifty 50 (IND)	4586	−15.644	< 0.001
MXX	Mexico Stock Exchange (MEX)	4640	−19.473	< 0.001
BVSP	Bovespa I. (BRA)	4556	−19.268	< 0.001
GSPTSE	Toronto Stock Exchange C. I. (CAN)	4043	−15.314	< 0.001
STOXX50E	Euro Stock 50 (EUR)	4713	−17.973	< 0.001
FTSTI	Straits Times I. (SGP)	2696	−16.045	< 0.001
FTSEMIB	Milano Indice di Borsa (ITA)	2307	−13.069	< 0.001

Figure: The ADF statistics and the p-values for the increments $Z_{t,1}$ of the Lamperti transform of $\log \sigma_t$ with $H = \overline{\hat{H}_0}$.

Data analysis, stationarity of transformed process

Ticker	ADF	p-value	Ticker	ADF	p-value	Ticker	ADF	p-value
SPX	-70.445	< 0.001	IXIC	-68.432	< 0.001	NSEI	-75.359	< 0.001
FTSE	-74.268	< 0.001	FCHI	-72.844	< 0.001	MXX	-74.510	< 0.001
N225	-69.858	< 0.001	HSI	-76.352	< 0.001	BVSP	-70.117	< 0.001
GDAXI	-74.407	< 0.001	KS11	-70.493	< 0.001	GSPTSE	-69.420	< 0.001
RUT	-72.623	< 0.001	AEX	-72.336	< 0.001	STOXX50E	$-77.253t$	< 0.001
AORD	-78.525	< 0.001	SSMI	-71.929	< 0.001	FTSTI	-58.243	< 0.001
DJI	-72.912	< 0.001	IBEX	-72.401	< 0.001	FTSEMIB	-50.403	< 0.001

Figure: The ADF statistics and the p-values for the increments $Z_{t,1}$ of the Lamperti transform of $\log \sigma_t$ with $H = \widehat{H}_0$, for each examined index.

Data analysis, autocorrelation

	1^{st} lag	2^{nd} lag	3^{rd} lag	4^{th} lag	5^{th} lag
SPX	-0.4017	-0.0142	-0.0339	0.0078	-0.0029
FTSE	-0.4336	-0.0118	-0.0308	-0.0007	0.0131
N225	-0.3715	-0.0565	-0.0018	-0.0421	0.0165
GDAXI	-0.3999	-0.0550	0.0127	-0.0513	0.0492
RUT	-0.4072	-0.0319	-0.0269	0.0137	-0.0105
AORD	-0.4802	0.0131	-0.0313	0.0419	-0.0412
DJI	-0.4338	0.0003	-0.0313	0.0146	-0.0121
IXIC	-0.3414	-0.0629	-0.0478	0.0178	-0.0054
FCHI	-0.3887	-0.0505	-0.0080	-0.0474	0.0756
HSI	-0.4620	-0.0020	0.0075	-0.0149	-0.0245
KS11	-0.4065	-0.0161	-0.0194	-0.0320	0.0019
AEX	-0.3855	-0.0483	-0.0124	-0.0453	0.0655
SSMI	-0.3863	-0.0501	-0.0054	-0.0223	0.0319
IBEX	-0.3753	-0.0650	-0.0079	-0.0351	0.0576
NSEI	-0.4111	-0.0567	0.0238	-0.0152	0.0129
MXX	-0.4528	0.0104	-0.0268	-0.0276	0.0245
BVSP	-0.3752	-0.0503	-0.0499	0.0003	0.0365
GSPTSE	-0.4527	0.0116	-0.0051	-0.0494	0.0160
STOXX50E	-0.4172	-0.0594	0.0202	-0.0308	0.0261
FTSTI	-0.4627	0.0072	-0.0306	0.0012	0.0002
FTSEMIB	-0.3764	-0.0573	-0.0070	-0.0528	0.0858
$fOU_{H=0.25}$	-0.2979	-0.0487	-0.0059	-0.0414	0.0072
$fOU_{H=0.50}$	-0.0143	-0.0096	-0.0061	-0.0186	0.0077
$fOU_{H=0.75}$	0.4084	0.2724	0.2234	0.1858	0.1816

Figure: First five-order autocorrelations for both indexes and fOU processes.

Data analysis, Regularity parameter I

Ticker	$\overline{\hat{H}_0}$	$\hat{\delta}$	p-value	Ticker	$\overline{\hat{H}_0}$	$\hat{\delta}$	p-value
SPX	0.142 ± 0.009	0.015 ± 0.001	0.67 ± 0.04	AEX	0.139 ± 0.014	0.020 ± 0.002	0.30 ± 0.10
FTSE	0.110 ± 0.011	0.015 ± 0.001	0.71 ± 0.09	SSMI	0.117 ± 0.008	0.014 ± 0.002	0.73 ± 0.11
N225	0.133 ± 0.008	0.011 ± 0.001	0.94 ± 0.04	IBEX	0.137 ± 0.025	0.024 ± 0.002	0.14 ± 0.05
GDAXI	0.133 ± 0.011	0.017 ± 0.001	0.49 ± 0.09	NSEI	0.130 ± 0.013	0.013 ± 0.007	0.86 ± 0.05
RUT	0.122 ± 0.008	0.012 ± 0.007	0.91 ± 0.03	MXX	0.091 ± 0.009	0.011 ± 0.001	0.95 ± 0.03
AORD	0.060 ± 0.010	0.008 ± 0.005	1.00 ± 0.00	BVSP	0.137 ± 0.015	0.015 ± 0.001	0.70 ± 0.10
DJI	0.112 ± 0.017	0.023 ± 0.001	0.16 ± 0.04	GSPTSE	0.087 ± 0.008	0.010 ± 0.001	0.97 ± 0.03
IXIC	0.151 ± 0.011	0.014 ± 0.002	0.75 ± 0.16	STOXX50E	0.147 ± 0.012	0.017 ± 0.001	0.53 ± 0.09
FCHI	0.117 ± 0.030	0.028 ± 0.002	0.05 ± 0.02	FTSTI	0.067 ± 0.011	0.025 ± 0.003	0.39 ± 0.16
HSI	0.071 ± 0.009	0.011 ± 0.001	0.93 ± 0.05	FTSEMIB	0.122 ± 0.021	0.035 ± 0.003	0.13 ± 0.05
KS11	0.113 ± 0.013	0.018 ± 0.002	0.49 ± 0.13				

Figure: Estimates $\overline{\hat{H}_0}$ (values averaged with respect to increasing upper timescale, $\underline{a} = 1$ and $\overline{a} \in [\![2, 20]\!]$). For AORD the exact p-value is 0.997 ± 0.003.

Data analysis, Regularity parameter II

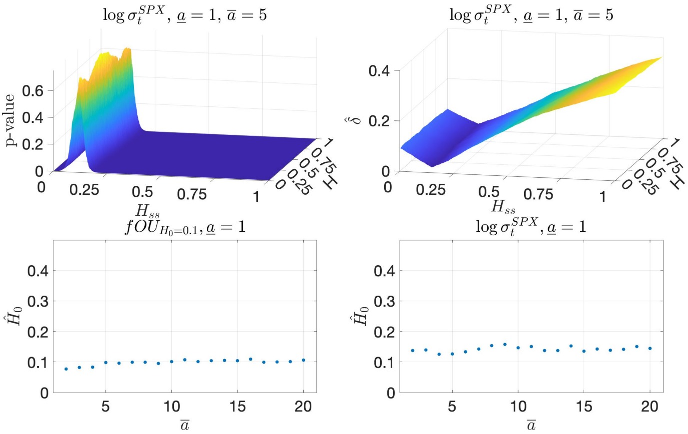

Figure: Top panels: table displays the distances $\hat{\delta}$ and the p-values for the KS tests about increments of Lamperti transform of $\log \sigma_t$ for SPX index with a set of timescales $\mathcal{A} = [1, 5]$. Bottom left: the $\hat{H}_0$ values estimated for a fOU with $H_0 = 0.10$ for different $\overline{a} \in [\![2, 20]\!]$. $\hat{H}_0$ are dispersed around the prescribed H_0 with a mean of $\overline{\hat{H}_0} = 0.098 \pm 0.009$. Bottom right: the same analysis is displayed for the $\log \sigma_t$ of the SPX index with an estimated Hurst exponent $\overline{\hat{H}_0} = 0.142 \pm 0.009$..

References I

Angelini, D. and S. Bianchi (2023). "Nonlinear biases in the roughness of a Fractional Stochastic Regularity Model". In: *Chaos, Solitons & Fractals* 172, p. 113550.

Bayer, C., P. Friz, and J. Gatheral (2016). "Pricing under rough volatility". In: *Quantitative Finance* 16(6), pp. 887–904.

Bennedsen, M., A. Lunde, and M.S. Pakkanen (Jan. 2021). "Decoupling the Short- and Long-Term Behavior of Stochastic Volatility". In: *Journal of Financial Econometrics* 20(5), pp. 961–1006.

Bianchi, S. (2004). "A New Distribution-Based Test of Self-similarity". In: *Fractals* 12(03), pp. 331–346. DOI: 10.1142/S0218348X04002586.

Bolko, E. et al. (2022). "A GMM approach to estimate the roughness of stochastic volatility". In: *Journal of Econometrics* Available online 15 September 2022.

References II

Cheridito, P., H. Kawaguchi, and M. Maejima (2003). “Fractional Ornstein-Uhlenbeck processes”. In: *Electronic Journal of Probability* 8, pp. 1–14.

Cont, R. and P. Das (Mar. 2022). “Rough Volatility: Fact or Artefact?” Available at SSRN: https://ssrn.com/abstract=4065951.

El Euch, O., J. Gatheral, and M. Rosenbaum (2019). “Roughening Heston”. In: *Risk, Available at SSRN: https://ssrn.com/abstract=3116887*, pp. 84–89.

Fukasawa, M., T. Takabatake, and R. Westphal (May 2019). *Is Volatility Rough?* Papers 1905.04852. arXiv.org.

Gatheral, J., T. Jaisson, and M. Rosenbaum (2014). “Volatility is rough”. Available at SSRN 2509457.

Gatheral, J., T. Jaisson, and M. Rosenbaum (2018). “Volatility is rough”. In: *Quantitative finance* 18(6), pp. 933–949.

References III

- Lamperti, J. (1962). "Semi-stable stochastic process". In: *Trans. Amer. Math. Soc.* 104, pp. 62–78.
- Livieri, G. et al. (2018). "Rough volatility: Evidence from option prices". In: *IISE Transactions* 50(9), pp. 767–776.
- Rogers, L.C.G. (2023). "Things We Think We Know, in Lecture Notes in Finance Options - 45 Years since the Publication of the Black-Scholes-Merton Model". In: World Scientific, pp. 173–184.
- Takaishi, T. (2020). "Rough volatility of Bitcoin". In: *Finance Research Letters* 32, p. 101379.